Caltech Harvey Mudd Math Competition from Fall 2010

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–	Team Round
1	In the diagram below, all circles are tangent to each other as shown. The six outer circles are all congruent to each other, and the six inner circles are all congruent to each other. Compute the ratio of the area of one of the outer circles to the area of one of the inner circles. `https://cdn.artofproblemsolving.com/attachments/b/6/4cfbc1df86b8d38e082b7ad0a71b9e366548b png`
2	Alfonso teaches Francis how to draw a spiral in the plane: First draw half of a unit circle. Starting at one of the ends, draw half a circle with radius $1/2$. Repeat this process at the endpoint of each half circle, where each time the radius is half of the previous half-circle. Assuming you can't stop Francis from drawing the entire spiral, compute the total length of the spiral.
3	In the diagram below, the three circles and the three line segments are tangent as shown. Given that the radius of all of the three circles is 1, compute the area of the triangle. `https://cdn.artofproblemsolving.com/attachments/b/e/8af4ea38d9a4c675edd0957aaa5336caec0ae png`
4	Dagan has a wooden cube. He paints each of the six faces a different color. He then cuts up the cube to get eight identically-sized smaller cubes, each of which now has three painted faces and three unpainted faces. He then puts the smaller cubes back together into one larger cube such that no unpainted face is visible. Compute the number of different cubes that Dagan can make this way. Two cubes are considered the same if one can be rotated to obtain the other. You may express your answer either as an integer or as a product of prime numbers.
5	The three positive integers a, b, c satisfy the equalities $gcd(ab, c^2) = 20$, $gcd(ac, b^2) = 18$, and $gcd(bc, a^2) = 75$. Compute the minimum possible value of $a + b + c$.
6	A 101×101 square grid is given with rows and columns numbered in order from 1 to 101. Each square that is contained in both an even-numbered row and an even-numbered column is cut out. A small section of the grid is shown below, with the cut-out squares in black. Compute the maximum number of L-triominoes (pictured below) that can be placed in the grid so that each L-triomino lies entirely inside the grid and no two overlap. Each L-triomino may be placed in the orientation pictured below, or rotated by 90^o, 180^o, or 270^o. `https://cdn.artofproblemsolving.com/attachments/2/5/016d4e823e3df4b83556a49f7e612d40e3deb png`

7 Art and Kimberly build flagpoles on a level ground with respective heights 10 m and 15 m, separated by a distance of 5 m. Kimberly wants to move her flagpole closer to Art's, but she can only doing so in the following manner:
1. Run a straight wire from the top of her flagpole to the bottom of Art's.
2. Run a straight wire from the top of Art's flagpole to the bottom of hers.
3. Build the flagpole to the point where the wires meet.
If Kimberly keeps moving her flagpole in this way, compute the number of flagpoles she will build whose heights are 1 m or greater (not counting her original 15 m flagpole).

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8 Rachel writes down a simple inequality: one 2-digit number is greater than another. Matt is sitting across from Rachel and peeking at her paper. If Matt, reading upside down, sees a valid inequality between two 2-digit numbers, compute the number of different inequalities that Rachel could have written. Assume that each digit is either a $1, 6, 8$, or 9.

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9 Let $a_0, a_1, ..., a_n$ be such that $a_n \neq 0$ and

$$(1 + x + x^3)^{342}(1 + 2x + x^2 + 2x^3 + 2x^4 + x^6)^{341} = \sum_{i=0}^{n} a_i x^i.$$

Compute the number of odd terms in the sequence $a_0, a_1, ..., a_n$.

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10 The 100th degree polynomial $P(x)$ satisfies $P(2^k) = k$ for $k = 0, 1, ...100$. Let a denote the leading coefficient of $P(x)$. Find the unique integer M such that $2^M < |a| < 2^{M+1}$.
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– Tiebreaker Round

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1 The numbers 25 and 76 have the property that when squared in base 10, their squares also end in the same two digits. A positive integer is called *amazing* if it has at most 3 digits when expressed in base 21 and also has the property that its square expressed in base 21 ends in the same 3 digits. (For this problem, the last three digits of a one-digit number b are 00b, and the last three digits of a two-digit number $\underline{ab}$ are $0\underline{ab}$.) Compute the sum of all amazing numbers. Express your answer in base 21.

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2 Let A, B, C, and D be points on a circle, in that order, such that $\overline{AD}$ is a diameter of the circle. Let E be the intersection of $\overleftrightarrow{AB}$ and $\overleftrightarrow{DC}$, let F be the intersection of $\overleftrightarrow{AC}$ and $\overleftrightarrow{BD}$, and let G be the intersection of $\overleftrightarrow{EF}$ and $\overleftrightarrow{AD}$. If $AD = 8$, $AE = 9$, and $DE = 7$, compute EG.

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3 Talithia throws a party on the fifth Saturday of every month that has five Saturdays. That is, if a month has five Saturdays, Talithia has a party on the fifth Saturday of that month, and if a month has four Saturdays, then Talithia does not have a party that month. Given that January $1, 2010$ was a Friday, compute the number of parties Talithia will have in 2010.

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4 Suppose a is a real number such that $3a + 6$ is the greatest integer less than or equal to a and $4a + 9$ is the least integer greater than or equal to a. Compute a.

– Mixer Round

Mixer *In this round, problems will depend on the answers to other problems. A bolded letter is used to denote a quantity whose value is determined by another problem's answer.*

Part I

p1. Let F be the answer to problem number 6.
You want to tile a nondegenerate square with side length F with 1×2 rectangles and 1×1 squares. The rectangles can be oriented in either direction. How many ways can you do this?

p2. Let **A** be the answer to problem number 1.
Triangle ABC has a right angle at B and the length of AC is **A**. Let D be the midpoint of AB, and let P be a point inside triangle ABC such that $PA = PC = \frac{7\sqrt{5}}{4}$ and $PD = \frac{7}{4}$. The length of AB^2 is expressible as m/n for m, n relatively prime positive integers. Find m.

p3. Let **B** be the answer to problem number 2.
Let S be the set of positive integers less than or equal to **B**. What is the maximum size of a subset of S whose elements are pairwise relatively prime?

p4. Let **C** be the answer to problem number 3.
You have 9 shirts and 9 pairs of pants. Each is either red or blue, you have more red shirts than blue shirts, and you have same number of red shirts as blue pants. Given that you have **C** ways of wearing a shirt and pants whose colors match, find out how many red shirts you own.

p5. Let **D** be the answer to problem number 4.
You have two odd positive integers a, b. It turns out that $lcm(a, b) + a = gcd(a, b) + b =$ **D**. Find ab.

p6. Let **E** be the answer to problem number 5.
A function f defined on integers satisfies $f(y) + f(12 - y) = 10$ and $f(y) + f(8 - y) = 4$ for all integers y. Given that $f($ **E** $) = 0$, compute $f(4)$.

Part II

p7. Let **L** be the answer to problem number 12.

You want to tile a nondegenerate square with side length **L** with 1×2 rectangles and 7×7 squares. The rectangles can be oriented in either direction. How many ways can you do this?

p8. Let **G** be the answer to problem number 7.
Triangle ABC has a right angle at B and the length of AC is **G**. Let D be the midpoint of AB, and let P be a point inside triangle ABC such that $PA = PC = \frac{1}{2}$ and $PD = \frac{1}{2010}$. The length of AB^2 is expressible as m/n for m, n relatively prime positive integers. Find m.

p9. Let **H** be the answer to problem number 8.
Let S be the set of positive integers less than or equal to **H**. What is the maximum size of a subset of S whose elements are pairwise relatively prime?

p10. Let **I** be the answer to problem number 9.
You have 391 shirts and 391 pairs of pants. Each is either red or blue, you have more red shirts than blue shirts, and you have same number of red shirts as red pants. Given that you have **I** ways of wearing a shirt and pants whose colors match, find out how many red shirts you own.

p11. Let **J** be the answer to problem number 10.
You have two odd positive integers a, b. It turns out that $lcm(a, b) + 2a = 2gcd(a, b) + b =$ **J**. Find ab.

p12. Let **K** be the answer to problem number 11.
A function f defined on integers satisfies $f(y) + f(7 - y) = 8$ and $f(y) + f(5 - y) = 4$ for all integers y. Given that $f($ **K** $) = 453$, compute $f(2)$.

PS. You had better use hide for answers. Collected here (`https://artofproblemsolving.com/community/c5h2760506p24143309`).

– Individual Round

1 Susan plays a game in which she rolls two fair standard six-sided dice with sides labeled one through six. She wins if the number on one of the dice is three times the number on the other die. If Susan plays this game three times, compute the probability that she wins at least once.

2 In triangles ABC and DEF, DE = 4AB, EF = 4BC, and F D = 4CA. The area of DEF is 360 units more than the area of ABC. Compute the area of ABC.

3 Andy has 2010 square tiles, each of which has a side length of one unit. He plans to arrange the tiles in an m x n rectangle, where mn = 2010. Compute the sum of the perimeters of all of the different possible rectangles he can make. Two rectangles are considered to be the same

if one can be rotated to become the other, so, for instance, a 1 x 2010 rectangle is considered to be the same as a 2010 x 1 rectangle.

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5 Let A and B be fixed points in the plane with distance AB = 1. An ant walks on a straight line from point A to some point C in the plane and notices that the distance from itself to B always decreases at any time during this walk. Compute the area of the region in the plane containing all points where point C could possibly be located.

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11 Darryl has a six-sided die with faces $1, 2, 3, 4, 5, 6$. He knows the die is weighted so that one face comes up with probability $1/2$ and the other five faces have equal probability of coming up. He unfortunately does not know which side is weighted, but he knows each face is equally likely to be the weighted one. He rolls the die 5 times and gets a $1, 2, 3, 4$ and 5 in some unspecified order. Compute the probability that his next roll is a 6.

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14 A 4-dimensional hypercube of edge length 1 is constructed in 4-space with its edges parallel to the coordinate axes and one vertex at the origin. The coordinates of its sixteen vertices are given
by (a, b, c, d), where each of $a, b, c,$ and d is either 0 or 1. The 3-dimensional hyperplane given by $x + y + z + w = 2$ intersects the hypercube at 6 of its vertices. Compute the 3-dimensional volume of the solid formed by the intersection.

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15 A student puts 2010 red balls and 1957 blue balls into a box. Weiqing draws randomly from the box one ball at a time without replacement. She wins if, at anytime, the total number of blue balls drawn is more than the total number of red balls drawn. Assuming Weiqing keeps drawing balls until she either wins or runs out, ompute the probability that she eventually wins.

– Individual Round

Individual **p1.** Susan plays a game in which she rolls two fair standard six-sided dice with sides labeled one through six. She wins if the number on one of the dice is three times the number on the other die. If Susan plays this game three times, compute the probability that she wins at least once.

p2. In triangles $\triangle\ ABC$ and $\triangle\ DEF$, $DE = 4AB$, $EF = 4BC$, and $FD = 4CA$. The area of $\triangle\ DEF$ is 360 units more than the area of $\triangle\ ABC$. Compute the area of $\triangle\ ABC$.

p3. Andy has 2010 square tiles, each of which has a side length of one unit. He plans to arrange the tiles in an $m \times n$ rectangle, where $mn = 2010$. Compute the sum of the perimeters of all of the different possible rectangles he can make. Two rectangles are considered to be the same if one can be rotated to become the other, so, for instance, a 1×2010 rectangle is considered to be the same as a 2010×1 rectangle.

p4. Let
$$S = \log_2 9 \log_3 16 \log_4 25 ... \log_{999} 1000000.$$
Compute the greatest integer less than or equal to $\log_2 S$.

p5. Let A and B be fixed points in the plane with distance $AB = 1$. An ant walks on a straight line from point A to some point C in the plane and notices that the distance from itself to B always decreases at any time during this walk. Compute the area of the region in the plane containing all points where point C could possibly be located.

p6. Lisette notices that $2^{10} = 1024$ and $2^{20} = 1048576$. Based on these facts, she claims that every number of the form 2^{10k} begins with the digit 1, where k is a positive integer. Compute the smallest k such that Lisette's claim is false. You may or may not find it helpful to know that $ln2 \approx 0.69314718$, $ln5 \approx 1.60943791$, and $log_{10}2 \approx 0 : 30103000$.

p7. Let S be the set of all positive integers relatively prime to 6. Find the value of $\sum_{k\in S} \frac{1}{2^k}$.

p8. Euclid's algorithm is a way of computing the greatest common divisor of two positive integers a and b with $a > b$. The algorithm works by writing a sequence of pairs of integers as follows.
1. Write down (a, b).
2. Look at the last pair of integers you wrote down, and call it (c, d). • If $d \ne 0$, let r be the remainder when c is divided by d. Write down (d, r). • If $d = 0$, then write down c. Once this happens, you're done, and the number you just wrote down is the greatest common divisor of a and b.
3. Repeat step 2 until you're done.
For example, with $a = 7$ and $b = 4$, Euclid's algorithm computes the greatest common divisor in 4 steps:
$$(7, 4) \to (4, 3) \to (3, 1) \to (1, 0) \to 1$$
For $a > b > 0,$ compute the least value of a such that Euclid's algorithm takes 10 steps to compute the greatest common divisor of a and b.

p9. Let $ABCD$ be a square of unit side length. Inscribe a circle C_0 tangent to all of the sides of the square. For each positive integer n, draw a circle Cn that is externally tangent to C_{n-1} and also tangent to sides AB and AD. Suppose r_i is the radius of circle C_i for every nonnegative integer i. Compute $\sqrt[200]{r_0/r_{100}}$.

p10. Rachel and Mike are playing a game. They start at 0 on the number line. At each positive integer on the number line, there is a carrot. At the beginning of the game, Mike picks a positive integer n other than 30. Every minute, Rachel moves to the next multiple of 30 on the number line that has a carrot on it and eats that carrot. At the same time, every minute, Mike moves to the next multiple of n on the number line that has a carrot on it and eats that carrot. Mike wants to pick an n such that, as the game goes on, he is always within 1000 units of Rachel. Compute the average (arithmetic mean) of all such n.

p11. Darryl has a six-sided die with faces $1, 2, 3, 4, 5, 6$. He knows the die is weighted so that one face comes up with probability $1/2$ and the other five faces have equal probability of coming up. He unfortunately does not know which side is weighted, but he knows each face is equally likely to be the weighted one. He rolls the die 5 times and gets a 1, 2, 3, 4 and 5 in some unspecified order. Compute the probability that his next roll is a 6.

p12. Let $F_0 = 1$, $F_1 = 1$ and $F_k = F_{k-1} + F_{k-2}$. Let $P(x) = \sum_{k=0}^{99} x^{F_k}$. The remainder when $P(x)$ is divided by $x^3 - 1$ can be expressed as $ax^2 + bx + c$. Find $2a + b$.

p13. Let $\theta \ne 0$ be the smallest acute angle for which $\sin\theta$, $\sin(2\theta)$, $\sin(3\theta)$, when sorted in increasing order, form an arithmetic progression. Compute $\cos(\theta/2)$.

p14. A 4-dimensional hypercube of edge length 1 is constructed in 4-space with its edges parallel to the coordinate axes and one vertex at the origin. The coordinates of its sixteen vertices are given by (a, b, c, d), where each of a, b, c, and d is either 0 or 1. The 3-dimensional hyperplane given by $x + y + z + w = 2$ intersects the hypercube at 6 of its vertices. Compute the 3-dimensional volume of the solid formed by the intersection.

p15. A student puts 2010 red balls and 1957 blue balls into a box. Weiqing draws randomly from the box one ball at a time without replacement. She wins if, at anytime, the total number of blue balls drawn is more than the total number of red balls drawn. Assuming Weiqing keeps drawing balls until she either wins or runs out, ompute the probability that she eventually wins.

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