



Caltech Harvey Mudd Math Competition from Spring 2012

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by parmenides51

– Team Round

1 Let a, b, c be positive integers. Suppose that $(a + b)(a + c) = 77$ and $(a + b)(b + c) = 56$. Find $(a + c)(b + c)$.

2 In the diagram below, A and B trisect DE , C and A trisect FG , and B and C trisect HI . Given that $DI = 5$, $EF = 6$, $GH = 7$, find the area of $\triangle ABC$.

<https://cdn.artofproblemsolving.com/attachments/d/5/90334e1bf62c99433be41f3b5e03c47c4d491.png>

3 In a 4×4 grid of sixteen unit squares, exactly 8 are shaded so that each shaded square shares an edge with exactly one other shaded square. How many ways can this be done?

4 Let $P(x)$ be a monic polynomial of degree 3. Suppose that $P(x)$ has remainder $R(x)$ when it is divided by $(x - 1)(x - 4)$ and $2R(x)$ when it is divided by $(x - 2)(x - 3)$. Given that $P(0) = 5$, find $P(5)$.

5 Suppose S is a subset of $\{1, 2, 3, 4, 5, 6, 7\}$. How many different possible values are there for the product of the elements in S ?

6 Compute

$$\prod_{k=1}^{12} \left(\prod_{j=1}^{10} \left(e^{2\pi j i / 11} - e^{2\pi k i / 13} \right) \right).$$

(The notation $\prod_{k=a}^b f(k)$ means the product $f(a)f(a + 1)\dots f(b)$.)

7 A positive integer x is k -equivocal if there exists two positive integers b, b' such that when x is represented in base b and base b' , the two representations have digit sequences of length k that are permutations of each other. The smallest 2-equivocal number is 7, since 7 is 21 in base 3 and 12 in base 5. Find the smallest 3-equivocal number.

8 A special kind of chess knight is in the origin of an infinite grid. It can make one of twelve different moves: it can move directly up, down, left, or right one unit square, or it can move 1 units in one direction and 3 units in an orthogonal direction. How many different squares can it be on after 2 moves?

- 9 Let S be a square of side length 1, one of whose vertices is A . Let S^+ be the square obtained by rotating S clockwise about A by 30° . Let S^- be the square obtained by rotating S counter-clockwise about A by 30° . Compute the total area that is covered by exactly two of the squares S, S^+, S^- . Express your answer in the form $a + b\sqrt{3}$ where a, b are rational numbers.

- 10 A convex polygon in the Cartesian plane has all of its vertices on integer coordinates. One of the sides of the polygon is AB where $A = (0, 0)$ and $B = (51, 51)$, and the interior angles at A and B are both at most 45 degrees. Assuming no 180 degree angles, what is the maximum number of vertices this polygon can have?

– Tiebreaker Round

- 1 Let a_k be the number of ordered 10-tuples $(x_1, x_2, \dots, x_{10})$ of nonnegative integers such that

$$x_1^2 + x_2^2 + \dots + x_{10}^2 = k.$$

Let $b_k = 0$ if a_k is even and $b_k = 1$ if a_k is odd. Find $\sum_{i=1}^{2012} b_{4i}$.

- 2 A convex octahedron in Cartesian space contains the origin in its interior. Two of its vertices are on the x -axis, two are on the y -axis, and two are on the z -axis. One triangular face F has side lengths $\sqrt{17}, \sqrt{37}, \sqrt{52}$. A second triangular face F_0 has side lengths $\sqrt{13}, \sqrt{29}, \sqrt{34}$. What is the minimum possible volume of the octahedron?

- 3 Three different faces of a regular dodecahedron are selected at random and painted. What is the probability that there is at least one pair of painted faces that share an edge?

- 4 The expression below has six empty boxes. Each box is to be filled in with a number from 1 to 6, where all six numbers are used exactly once, and then the expression is evaluated. What is the maximum possible final result that can be achieved?

$$\frac{\square + \square}{\square}$$

– Mixer Round

Mixer Part 1

You might think this round is broken after solving some of these problems, but everything is intentional.

- 1.1. The number n can be represented uniquely as the sum of 6 distinct positive integers. Find n .

1.2. Let ABC be a triangle with $AB = BC$. The altitude from A intersects line BC at D . Suppose $BD = 5$ and $AC^2 = 1188$. Find AB .

1.3. A lemonade stand analyzes its earning and operations. For the previous month it had a \$45 dollar budget to divide between production and advertising. If it spent k dollars on production, it could make $2k - 12$ glasses of lemonade. If it spent k dollars on advertising, it could sell each glass at an average price of $15 + 5k$ cents. The amount it made in sales for the previous month was \$40.50. Assuming the stand spent its entire budget on production and advertising, what was the absolute difference between the amount spent on production and the amount spent on advertising?

1.4. Let A be the number of different ways to tile a $1 \times n$ rectangle with tiles of size 1×1 , 1×3 , and 1×6 . Let B be the number of different ways to tile a $1 \times n$ rectangle with tiles of size 1×2 and 1×5 , where there are 2 different colors available for the 1×2 tiles. Given that $A = B$, find n . (Two tilings that are rotations or reflections of each other are considered distinct.)

1.5. An integer $n \geq 0$ is such that n when represented in base 2 is written the same way as $2n$ is in base 5. Find n .

1.6. Let x be a positive integer such that $3, \log_6(12x), \log_6(18x)$ form an arithmetic progression in some order. Find x .

Part 2

Oops, it looks like there were some *intentional* printing errors and some of the numbers from these problems got removed. Any ■ that you see was originally some positive integer, but now its value is no longer readable. Still, if things behave like they did for Part 1, maybe you can piece the answers together.

2.1. The number n can be represented uniquely as the sum of ■ distinct positive integers. Find n .

2.2. Let ABC be a triangle with $AB = BC$. The altitude from A intersects line BC at D . Suppose $BD = \blacksquare$ and $AC^2 = 1536$. Find AB .

2.3. A lemonade stand analyzes its earning and operations. For the previous month it had a \$50 dollar budget to divide between production and advertising. If it spent k dollars on production, it could make $2k - 2$ glasses of lemonade. If it spent k dollars on advertising, it could sell each glass at an average price of $25 + 5k$ cents. The amount it made in sales for the previous

month was \$ \blacksquare . Assuming the stand spent its entire budget on production and advertising, what was the absolute difference between the amount spent on production and the amount spent on advertising?

2.4. Let A be the number of different ways to tile a $1 \times n$ rectangle with tiles of size $1 \times \blacksquare$, $1 \times \blacksquare$, and $1 \times \blacksquare$. Let B be the number of different ways to tile a $1 \times n$ rectangle with tiles of size $1 \times \blacksquare$ and $1 \times \blacksquare$, where there are \blacksquare different colors available for the $1 \times \blacksquare$ tiles. Given that $A = B$, find n . (Two tilings that are rotations or reflections of each other are considered distinct.)

2.5. An integer $n \geq \blacksquare$ is such that n when represented in base 9 is written the same way as $2n$ is in base \blacksquare . Find n .

2.6. Let x be a positive integer such that $1, \log_{96}(6x), \log_{96}(\blacksquare x)$ form an arithmetic progression in some order. Find x .

PS. You had better use hide for answers. Collected here (<https://artofproblemsolving.com/community/c5h2760506p24143309>).

– Individual Round

Individual p1. A robot is at position 0 on a number line. Each second, it randomly moves either one unit in the positive direction or one unit in the negative direction, with probability $\frac{1}{2}$ of doing each. Find the probability that after 4 seconds, the robot has returned to position 0.

p2. How many positive integers $n \leq 20$ are such that the greatest common divisor of n and 20 is a prime number?

p3. A sequence of points $A_1, A_2, A_3, \dots, A_7$ is shown in the diagram below, with A_1A_2 parallel to A_6A_7 . We have $\angle A_2A_3A_4 = 113^\circ$, $\angle A_3A_4A_5 = 100^\circ$, and $\angle A_4A_5A_6 = 122^\circ$. Find the degree measure of $\angle A_1A_2A_3 + \angle A_5A_6A_7$.

<https://cdn.artofproblemsolving.com/attachments/d/a/75b06a6663b2f4258e35ef0f68fcfbfaa903f7.png>

p4. Compute

$$\log_3 \left(\frac{\log_3 3^{3^{3^3}}}{\log_{3^3} 3^{3^3}} \right)$$

p5. In an 8×8 chessboard, a pawn has been placed on the third column and fourth row, and all the other squares are empty. It is possible to place nine rooks on this board such that no two

rooks attack each other. How many ways can this be done? (Recall that a rook can attack any square in its row or column provided all the squares in between are empty.)

p6. Suppose that a, b are positive real numbers with $a > b$ and $ab = 8$. Find the minimum value of $\frac{a^2+b^2}{a-b}$.

p7. A cone of radius 4 and height 7 has A as its apex and B as the center of its base. A second cone of radius 3 and height 7 has B as its apex and A as the center of its base. What is the volume of the region contained in both cones?

p8. Let $a_1, a_2, a_3, a_4, a_5, a_6$ be a permutation of the numbers 1, 2, 3, 4, 5, 6. We say a_i is visible if a_i is greater than any number that comes before it; that is, $a_j < a_i$ for all $j < i$. For example, the permutation 2, 4, 1, 3, 6, 5 has three visible elements: 2, 4, 6. How many such permutations have exactly two visible elements?

p9. Let $f(x) = x + 2x^2 + 3x^3 + 4x^4 + 5x^5 + 6x^6$, and let $S = [f(6)]^5 + [f(10)]^3 + [f(15)]^2$. Compute the remainder when S is divided by 30.

p10. In triangle ABC , the angle bisector from A and the perpendicular bisector of BC meet at point D , the angle bisector from B and the perpendicular bisector of AC meet at point E , and the perpendicular bisectors of BC and AC meet at point F . Given that $\angle ADF = 5^\circ$, $\angle BEF = 10^\circ$, and $AC = 3$, find the length of DF .

<https://cdn.artofproblemsolving.com/attachments/6/d/6bb8409678a4c44135d393b9b942f8defb198.png>

p11. Let $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$. How many subsets S of $\{1, 2, \dots, 2011\}$ are there such that

$$F_{2012} - 1 = \sum_{i \in S} F_i?$$

p12. Let a_k be the number of perfect squares m such that $k^3 \leq m < (k+1)^3$. For example, $a_2 = 3$ since three squares m satisfy $2^3 \leq m < 3^3$, namely 9, 16, and 25. Compute

$$\sum_{k=0}^{99} \lfloor \sqrt{k} \rfloor a_k,$$

where $\lfloor x \rfloor$ denotes the largest integer less than or equal to x .

p13. Suppose that a, b, c, d, e, f are real numbers such that

$$a + b + c + d + e + f = 0,$$

$$a + 2b + 3c + 4d + 2e + 2f = 0,$$

$$a + 3b + 6c + 9d + 4e + 6f = 0,$$

$$a + 4b + 10c + 16d + 8e + 24f = 0,$$

$$a + 5b + 15c + 25d + 16e + 120f = 42.$$

Compute $a + 6b + 21c + 36d + 32e + 720f$.

p14. In Cartesian space, three spheres centered at $(-2, 5, 4)$, $(2, 1, 4)$, and $(4, 7, 5)$ are all tangent to the xy -plane. The xy -plane is one of two planes tangent to all three spheres; the second plane can be written as the equation $ax + by + cz = d$ for some real numbers a, b, c, d . Find $\frac{c}{a}$.

p15. Find the number of pairs of positive integers a, b , with $a \leq 125$ and $b \leq 100$, such that $a^b - 1$ is divisible by 125.

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