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Art of Problem Solving

Caltech Harvey Mudd Math Competition from Fall 2012

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by parmenides51

–	Team Round
1	Find the remainder when 5^{2012} is divided by 3.
2	Consider a triangle ABC with points D on AB, E on BC, and let F be the intersection of AE and CD. Suppose $AD = 1$, $DB = 2$,$BE = 1$,$EC = 3$, and $CA = 5$. Find the value of the area of ECF minus the area of ADF.
3	A particular graph has 6 vertices, 12 edges, and has the property that it contains no Eulerian path; a Eulerian path is a route from vertex to vertex along edges that traces each edge exactly once. Determine all the possible degrees of its vertices in no particular order. There are two solutions, and you need to get both to get credit for this problem.

4	Consider the figure below, not drawn to scale.
	In this figure, assume that$AB \perp BE$ and $AD \perp DE$. Also, let $AB = \sqrt{6}$ and $\angle BED = \frac{\pi}{6}$. Find AC.
	`https://cdn.artofproblemsolving.com/attachments/2/d/f87ac9f111f02e261a0b5376c766a615e8d1c`
	`png`

5	At each step, a rectangular tile of length $1, 2$, or, 3 is chosen at random, what is the probability that the total length is 10 after 5 steps?
6	Suppose you have ten pairs of red socks, ten pairs of blue socks, and ten pairs of green socks in your drawer. You need to go to a party soon, but the power is currently off in your room. It is completely dark, so you cannot see any colors and unfortunately the socks are identically shaped. What is the minimum number of socks you need to take from the drawer in order to guarantee that you have at least one pair of socks whose colors match?

7	Consider a 1 by 2 by 3 rectangular prism. Find the length of the shortest path between opposite corners A and B that does not leave the surface of the prism.
	`https://cdn.artofproblemsolving.com/attachments/7/1/899b79b0f938e63fe7b093cecff63ed1255ac`
	`png`

8	Find the sum of all positive 30-digit palindromes. The leading digit is not allowed to be 0.
9	For a positive integer n, let $f(n)$ be equal to n if there is an integer x such that $x^2 - n$ is divisible

by 2^{12}, and let $f(n)$ be 0 otherwise. Determine the remainder when

$$\sum_{n=0}^{2^{12}-1} f(n)$$

is divided by 2^{12}.

10 Let

$$N = \binom{2^{2012}}{0}\binom{2^{2012}}{1}\binom{2^{2012}}{2}\binom{2^{2012}}{3}\cdots\binom{2^{2012}}{2^{2012}}.$$

Let M be the number of 0's when N is written in binary. How many 0's does M have when written in binary?
(Warning: this question is very hard.)

– Tiebreaker Round

1 Let $[n] = \{1, 2, 3, ..., n\}$ and for any set S, let$P(S)$ be the set of non-empty subsets of S. What is the last digit of $|P(P([2013]))|$?

2 Find all continuous functions $f : R \to R$ such that

$$f(x + f(y)) = f(x + y) + y,$$

for all $x, y \in R$. No proof is required for this problem.

3 For a positive integer n, let $\sigma(n)$ be the sum of the divisors of n (for example $\sigma(10) = 1 + 2 + 5 + 10 = 18$). For how many $n \in \{1, 2, ..., 100\}$, do we have $\sigma(n) < n + \sqrt{n}$?

4 A lattice point $(x, y, z) \in Z^3$ can be seen from the origin if the line from the origin does not contain any other lattice point (x', y', z') with

$$(x')^2 + (y')^2 + (z')^2 < x^2 + y^2 + z^2.$$

Let p be the probability that a randomly selected point on the cubic lattice Z^3 can be seen from the origin. Given that

$$\frac{1}{p} = \sum_{n=i}^{\infty} \frac{k}{n^s}$$

for some integers i, k, and s, find i, k and s.

– Mixer Round

Mixer **p1.** Prove that $x = 2$ is the only real number satisfying $3^x + 4^x = 5^x$.

p2. Show that $\sqrt{9+4\sqrt5}-\sqrt{9-4\sqrt5}$ is an integer.

p3. Two players A and B play a game on a round table. Each time they take turn placing a round coin on the table. The coin has a uniform size, and this size is at least 10 times smaller than the table size. They cannot place the coin on top of any part of other coins, and the whole coin must be on the table. If a player cannot place a coin, he loses. Suppose A starts first. If both of them plan their moves wisely, there will be one person who will always win. Determine whether A or B will win, and then determine his winning strategy.

p4. Suppose you are given 4 pegs arranged in a square on a board. A ”move” consists of picking up a peg, reflecting it through any other peg, and placing it down on the board. For how many integers $1 \le n \le 2013$ is it possible to arrange the 4 pegs into a *larger* square using exactly n moves? Justify your answers.

p5. Find smallest positive integer that has a remainder of 1 when divided by 2, a remainder of 2 when divided by 3, a remainder of 3 when divided by 5, and a remainder of 5 when divided by 7.

p6. Find the value of
$$\sum_{m|496,m>0}\frac{1}{m},$$
where $m|496$ means 496 is divisible by m.

p7. What is the value of
$$\binom{100}{0}+\binom{100}{4}+\binom{100}{8}+...+\binom{100}{100}?$$

p8. An n-term sequence $a_0, a_1, ..., a_n$ will be called *sweet* if, for each $0 \le i \le n-1$, a_i is the number of times that the number i appears in the sequence. For example, $1, 2, 1, 0$ is a sweet sequence with 4 terms. Given that a_0, a_1, ..., a_{2013} is a sweet sequence, find the value of $a_0^2 + a_1^2 + ... + a_{2013}^2$.

PS. You had better use hide for answers. Collected here (`https://artofproblemsolving.com/community/c5h2760506p24143309`).

– Individual Round

Individual **p1.** How many nonzero digits are in the number $(5^{94} + 5^{92})(2^{94} + 2^{92})$?

p2. Suppose A is a set of 2013 distinct positive integers such that the arithmetic mean of any subset of A is also an integer. Find an example of A.

p3. How many minutes until the smaller angle formed by the minute and hour hands on the face of a clock is congruent to the smaller angle between the hands at $5 : 15$ pm? Round your answer to the nearest minute.

p4. Suppose a and b are positive real numbers, $a + b = 1$, and

$$1 + \frac{a^2 + 3b^2}{2ab} = \sqrt{4 + \frac{a}{b} + \frac{3b}{a}}.$$

Find a.

p5. Suppose $f(x) = \frac{e^x - 12e^{-x}}{2}$. Find all x such that $f(x) = 2$.

p6. Let P_1, P_2,...,P_n be points equally spaced on a unit circle. For how many integer $n \in \{2, 3, ..., 2013\}$ is the product of all pairwise distances: $\prod_{1\le i<j\le n} P_iP_j$ a rational number?
Note that $\prod$ means the product. For example, $\prod_{1\le i\le 3} i = 1 \cdot 2 \cdot 3 = 6.$

p7. Determine the value a such that the following sum converges if and only if $r \in (-\infty, a)$:

$$\sum_{n=1}^{\infty}(\sqrt{n^4 + n^r} - n^2).$$

Note that $\sum_{n=1}^{\infty}\frac{1}{n^s}$ converges if and only if $s > 1$.

p8. Find two pairs of positive integers (a, b) with $a > b$ such that $a^2 + b^2 = 40501.$

p9. Consider a simplified memory-knowledge model. Suppose your total knowledge level the night before you went to a college was 100 units. Each day, when you woke up in the morning you forgot 1% of what you had learned. Then, by going to lectures, working on the homework, preparing for presentations, you had learned more and so your knowledge level went up by 10 units at the end of the day.
According to this model, how long do you need to stay in college until you reach the knowledge level of exactly 1000?

p10. Suppose $P(x) = 2x^8 + x^6 - x^4 + 1$, and that P has roots a_1, a_2, ... , a_8 (a complex number z is a root of the polynomial $P(x)$ if $P(z) = 0$). Find the value of

$$(a_1^2 - 2)(a_2^2 - 2)(a_3^2 - 2)...(a_8^2 - 2).$$

p11. Find all values of x satisfying $(x^2 + 2x - 5)^2 = -2x^2 - 3x + 15$.

p12. Suppose x, y and z are positive real numbers such that

$$x^2 + y^2 + xy = 9,$$
$$y^2 + z^2 + yz = 16,$$
$$x^2 + z^2 + xz = 25.$$

Find $xy + yz + xz$ (the answer is unique).

p13. Suppose that $P(x)$ is a monic polynomial (i.e, the leading coefficient is 1) with 20 roots, each distinct and of the form $\frac{1}{3^k}$ for $k = 0, 1, 2, ..., 19$. Find the coefficient of x^{18} in $P(x)$.

p14. Find the sum of the reciprocals of all perfect squares whose prime factorization contains only powers of 3, 5, 7 (i.e. $\frac{1}{1} + \frac{1}{9} + \frac{1}{25} + \frac{1}{419} + \frac{1}{811} + \frac{1}{215} + \frac{1}{441} + \frac{1}{625} + ...$).

p15. Find the number of integer quadruples (a, b, c, d) which also satisfy the following system of equations:

$$1 + b + c^2 + d^3 = 0,$$
$$a + b^2 + c^3 + d^4 = 0,$$
$$a^2 + b^3 + c^4 + d^5 = 0,$$
$$a^3 + b^4 + c^5 + d^6 = 0.$$

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