China Team Selection Test 2022

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by Tintarn, JustPostChinaTST

Test 1 Day 1

1 In a cyclic convex hexagon $ABCDEF$, AB and DC intersect at G, AF and DE intersect at H. Let M, N be the circumcenters of BCG and EFH, respectively. Prove that the BE, CF and MN are concurrent.

2 Let p be a prime, A is an infinite set of integers. Prove that there is a subset B of A with $2p-2$ elements, such that the arithmetic mean of any pairwise distinct p elements in B does not belong to A.

3 Let a, b, c, p, q, r be positive integers with $p, q, r \ge 2$. Denote

$$Q = \{(x, y, z) \in \mathbb{Z}^3 : 0 \le x \le a, 0 \le y \le b, 0 \le z \le c\}.$$

Initially, some pieces are put on the each point in Q, with a total of M pieces. Then, one can perform the following three types of operations repeatedly:
(1) Remove p pieces on (x, y, z) and place a piece on $(x-1, y, z)$;
(2) Remove q pieces on (x, y, z) and place a piece on $(x, y-1, z)$;
(3) Remove r pieces on (x, y, z) and place a piece on $(x, y, z-1)$.

Find the smallest positive integer M such that one can always perform a sequence of operations, making a piece placed on $(0, 0, 0)$, no matter how the pieces are distributed initially.

Test 1 Day 2

4 Let ABC be an acute triangle with $\angle ACB > 2\angle ABC$. Let I be the incenter of ABC, K is the reflection of I in line BC. Let line BA and KC intersect at D. The line through B parallel to CI intersects the minor arc BC on the circumcircle of ABC at $E(E \neq B)$. The line through A parallel to BC intersects the line BE at F.
Prove that if $BF = CE$, then $FK = AD$.

5 Let $C = \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle on the complex plane. Let $z_1, z_2, \ldots, z_{240} \in C$ (not necessarily different) be 240 complex numbers, satisfying the following two conditions:
(1) For any open arc Γ of length π on C, there are at most 200 of j $(1 \le j \le 240)$ such that $z_j \in \Gamma$.
(2) For any open arc γ of length $\pi/3$ on C, there are at most 120 of j $(1 \le j \le 240)$ such that $z_j \in \gamma$.

Find the maximum of $|z_1 + z_2 + \ldots + z_{240}|$.

6 Let m be a positive integer, and $A_1, A_2, \dots, A_m$ (not necessarily different) be m subsets of a finite set A. It is known that for any nonempty subset I of $\{1, 2 \dots, m\}$,

$$\left| \bigcup_{i \in I} A_i \right| \geq |I| + 1.$$

Show that the elements of A can be colored black and white, so that each of $A_1, A_2, \dots, A_m$ contains both black and white elements.

Test 2 Day 1

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1 Find all pairs of positive integers (m, n), such that in a $m \times n$ table (with $m+1$ horizontal lines and $n+1$ vertical lines), a diagonal can be drawn in some unit squares (some unit squares may have no diagonals drawn, but two diagonals cannot be both drawn in a unit square), so that the obtained graph has an Eulerian cycle.

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2 Given a non-right triangle ABC with $BC > AC > AB$. Two points $P_1 \neq P_2$ on the plane satisfy that, for $i = 1, 2$, if AP_i, BP_i and CP_i intersect the circumcircle of the triangle ABC at D_i, E_i, and F_i, respectively, then $D_iE_i \perp D_iF_i$ and $D_iE_i = D_iF_i \neq 0$. Let the line P_1P_2 intersects the circumcircle of ABC at Q_1 and Q_2. The Simson lines of Q_1, Q_2 with respect to ABC intersect at W.

Prove that W lies on the nine-point circle of ABC.

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3 Let $a_1, a_2, \dots, a_n$ be n positive integers that are not divisible by each other, i.e. for any $i \neq j$, a_i is not divisible by a_j. Show that

$$a_1 + a_2 + \cdots + a_n \geq 1.1n^2 - 2n.$$

Note: A proof of the inequality when n is sufficient large will be awarded points depending on your results.

Test 2 Day 2

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4 Given a positive integer n, find all n-tuples of real number $(x_1, x_2, \dots, x_n)$ such that

$$f(x_1, x_2, \cdots, x_n) = \sum_{k_1=0}^{2} \sum_{k_2=0}^{2} \cdots \sum_{k_n=0}^{2} \left| k_1x_1 + k_2x_2 + \cdots + k_nx_n - 1 \right|$$

attains its minimum.

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5 Given a positive integer n, let D is the set of positive divisors of n, and let $f : D \to \mathbb{Z}$ be a function. Prove that the following are equivalent:

(a) For any positive divisor m of n,

$$n \;\Big|\; \sum_{d|m} f(d)\binom{n/d}{m/d}.$$

(b) For any positive divisor k of n,

$$k \;\Big|\; \sum_{d|k} f(d).$$

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6 Let m, n be two positive integers with $m \geq n \geq 2022$. Let $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n$ be $2n$ real numbers. Prove that the numbers of ordered pairs (i, j) $(1 \leq i, j \leq n)$ such that

$$|a_i + b_j - ij| \leq m$$

does not exceed $3n\sqrt{m \log n}$.

Test 3 Day 1

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1 Given two circles ω_1 and ω_2 where ω_2 is inside ω_1. Show that there exists a point P such that for any line ℓ not passing through P, if ℓ intersects circle ω_1 at A, B and ℓ intersects circle ω_2 at C, D, where A, C, D, B lie on ℓ in this order, then $\angle APC = \angle BPD$.

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2 Two positive real numbers α, β satisfies that for any positive integers k_1, k_2, it holds that $\lfloor k_1\alpha \rfloor \neq \lfloor k_2\beta \rfloor$, where $\lfloor x \rfloor$ denotes the largest integer less than or equal to x. Prove that there exist positive integers m_1, m_2 such that $\frac{m_1}{\alpha} + \frac{m_2}{\beta} = 1$.

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3 Given a positive integer $n \geq 2$. Find all n-tuples of positive integers $(a_1, a_2, \dots, a_n)$, such that $1 < a_1 \leq a_2 \leq a_3 \leq \cdots \leq a_n$, a_1 is odd, and
(1) $M = \frac{1}{2^n}(a_1 - 1)a_2 a_3 \cdots a_n$ is a positive integer;
(2) One can pick n-tuples of integers $(k_{i,1}, k_{i,2}, \dots, k_{i,n})$ for $i = 1, 2, \dots, M$ such that for any $1 \leq i_1 < i_2 \leq M$, there exists $j \in \{1, 2, \dots, n\}$ such that $k_{i_1,j} - k_{i_2,j} \not\equiv 0, \pm 1 \pmod{a_j}$.

Test 3 Day 2

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4 Find all positive integer k such that one can find a number of triangles in the Cartesian plane, the centroid of each triangle is a lattice point, the union of these triangles is a square of side length k (the sides of the square are not necessarily parallel to the axis, the vertices of the square are not necessarily lattice points), and the intersection of any two triangles is an empty-set, a common point or a common edge.

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5 Show that there exist constants c and $\alpha > \frac{1}{2}$, such that for any positive integer n, there is a subset A of $\{1, 2, \dots, n\}$ with cardinality $|A| \geq c \cdot n^\alpha$, and for any $x, y \in A$ with $x \neq y$, the difference $x - y$ is not a perfect square.

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6 (1) Prove that, on the complex plane, the area of the convex hull of all complex roots of $z^{20}+63z+22=0$ is greater than π.
(2) Let $a_1, a_2, \dots, a_n$ be complex numbers with sum 1, and $k_1 < k_2 < \dots < k_n$ be odd positive integers. Let ω be a complex number with norm at least 1. Prove that the equation

$$a_1 z^{k_1} + a_2 z^{k_2} + \dots + a_n z^{k_n} = w$$

has at least one complex root with norm at most $3n|\omega|$.

Test 4 Day 1

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1 Initially, each unit square of an $n \times n$ grid is colored red, yellow or blue. In each round, perform the following operation for every unit square simultaneously:

- For a red square, if there is a yellow square that has a common edge with it, then color it yellow.
- For a yellow square, if there is a blue square that has a common edge with it, then color it blue.
- For a blue square, if there is a red square that has a common edge with it, then color it red.

It is known that after several rounds, all unit squares of this $n \times n$ grid have the same color. Prove that the grid has became monochromatic no later than the end of the $(2n-2)$-th round.

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2 Let $ABCD$ be a convex quadrilateral, the incenters of $\triangle ABC$ and $\triangle ADC$ are I, J, respectively. It is known that AC, BD, IJ concurrent at a point P. The line perpendicular to BD through P intersects with the outer angle bisector of $\angle BAD$ and the outer angle bisector $\angle BCD$ at E, F, respectively. Show that $PE = PF$.

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3 Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for any $x, y \in \mathbb{R}$, the multiset $\{(f(xf(y)+1), f(yf(x)-1)\}$ is identical to the multiset $\{xf(f(y))+1, yf(f(x))-1\}$.

Note: The multiset $\{a, b\}$ is identical to the multiset $\{c, d\}$ if and only if $a = c, b = d$ or $a = d, b = c$.

Test 4 Day 2

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4 Find all positive integers a, b, c and prime p satisfying that

$$2^a p^b = (p+2)^c + 1.$$

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5 Let n be a positive integer, $x_1, x_2, \dots, x_{2n}$ be non-negative real numbers with sum 4. Prove that there exist integer p and q, with $0 \le q \le n-1$, such that

$$\sum_{i=1}^{q} x_{p+2i-1} \le 1 \text{ and } \sum_{i=q+1}^{n-1} x_{p+2i} \le 1,$$

where the indices are take modulo $2n$.

Note: If $q=0$, then $\sum_{i=1}^{q} x_{p+2i-1}=0$; if $q=n-1$, then $\sum_{i=q+1}^{n-1} x_{p+2i}=0$.

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6 Given a positive integer n, let D be the set of all positive divisors of n. The subsets A, B of D satisfies that for any $a \in A$ and $b \in B$, it holds that $a \nmid b$ and $b \nmid a$. Show that

$$\sqrt{|A|}+\sqrt{|B|} \leq \sqrt{|D|}.$$
