Mid-Michigan Mathematical Olympiad, Grades 10-12
www.artofproblemsolving.com/community/c3014890
by parmenides51

2002 **p1.** Find all integer solutions of the equation $a^2 - b^2 = 2002$.

p2. Prove that the disks drawn on the sides of a convex quadrilateral as on diameters cover this quadrilateral.

p3. 30 students from one school came to Mathematical Olympiad. In how many different ways is it possible to place them in four rooms?

p4. A 12 liter container is filled with gasoline. How to split it in two equal parts using two empty 5 and 8 liter containers?

PS. You should use hide for answers. Collected here (`https://artofproblemsolving.com/community/c5h2760506p24143309`).

2003 **p1.** The length of the first side of a triangle is 1, the length of the second side is 11, and the length of the third side is an integer. Find that integer.

p2. Suppose a, b, and c are positive numbers such that $a+b+c = 1$. Prove that $ab+ac+bc \le \frac{1}{3}$.

p3. Prove that $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ... + \frac{1}{100}$ is not an integer.

p4. Find all of the four-digit numbers n such that the last four digits of n^2 coincide with the digits of n.

p5. (Bonus) Several ants are crawling along a circle with equal constant velocities (not necessarily in the same direction). If two ants collide, both immediately reverse direction and crawl with the same velocity. Prove that, no matter how many ants and what their initial positions are, they will, at some time, all simultaneously return to the initial positions.

PS. You should use hide for answers. Collected here (`https://artofproblemsolving.com/community/c5h2760506p24143309`).

2004 **p1.** Two players play the following game. On the lowest left square of an 8×8 chessboard there is a rook (castle). The first player is allowed to move the rook up or to the right by an arbitrary number of squares. The second layer is also allowed to move the rook up or to the right by an arbitrary number of squares. Then the first player is allowed to do this again, and so on. The one who moves the rook to the upper right square wins. Who has a winning strategy?

p2. Find the smallest positive whole number that ends with 17, is divisible by 17, and the sum of its digits is 17.

p3. Three consecutive 2-digit numbers are written next to each other. It turns out that the resulting 6-digit number is divisible by 17. Find all such numbers.

p4. Let $ABCD$ be a convex quadrilateral (a quadrilateral $ABCD$ is called convex if the diagonals AC and BD intersect). Suppose that $\angle CBD = \angle CAB$ and $\angle ACD = \angle BDA$. Prove that $\angle ABC = \angle ADC$.

p5. A circle of radius 1 is cut into four equal arcs, which are then arranged to make the shape shown on the picture. What is its area?
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PS. You should use hide for answers. Collected here (https://artofproblemsolving.com/community/c5h2760506p24143309).

2005 **p1.** A tennis net is made of strings tied up together which make a grid consisting of small squares as shown below.
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png
The size of the net is 100×10 small squares. What is the maximal number of sides of small squares which can be cut without breaking the net into two separate pieces? (The side is cut only in the middle, not at the ends).

p2. What number is bigger 2^{300} or 3^{200} ?

p3. All noble knights participating in a medieval tournament in Camelot used nicknames. In the tournament each knight had combats with all other knights. In each combat one knight won and the second one lost. At the end of tournament the losers reported their real names to the winners and to the winners of their winners. Was there a person who knew the real names of all knights?

p4. Two players Tom and Sid play the following game. There are two piles of rocks, 10 rocks in the first pile and 12 rocks in the second pile. Each of the players in his turn can take either any amount of rocks from one pile or the same amount of rocks from both piles. The winner is the player who takes the last rock. Who does win in this game if Tom starts the game?

p5. There is an interesting 5-digit integer. With a 1 after it, it is three times as large as with a 1 before it. What is the number?

PS. You should use hide for answers. Collected here (`https://artofproblemsolving.com/community/c5h2760506p24143309`).

2006 **p1.** A right triangle has hypotenuse of length 12 cm. The height corresponding to the right angle has length 7 cm. Is this possible?
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`png`

p2. Prove that from any 5 integers one can choose 3 such that their sum is divisible by 3.

p3. Two players play the following game on an 8×8 chessboard. The first player can put a knight on an arbitrary square. Then the second player can put another knight on a free square that is not controlled by the first knight. Then the first player can put a new knight on a free square that is not controlled by the knights on the board. Then the second player can do the same, etc. A player who cannot put a new knight on the board loses the game. Who has a winning strategy?

p4. Consider a regular octagon $ABCDEGH$ (i.e., all sides of the octagon are equal and all angles of the octagon are equal). Show that the area of the rectangle $ABEF$ is one half of the area of the octagon.
`https://cdn.artofproblemsolving.com/attachments/d/1/674034f0b045c0bcde3d03172b01aae337fba`
`png`

p5. Can you find a positive whole number such that after deleting the first digit and the zeros following it (if they are) the number becomes 24 times smaller?

PS. You should use hide for answers. Collected here (`https://artofproblemsolving.com/community/c5h2760506p24143309`).

2007 **p1.** 17 rooks are placed on an 8×8 chess board. Prove that there must be at least one rook that is attacking at least 2 other rooks.

p2. In New Scotland there are three kinds of coins: 1 cent, 6 cent, and 36 cent coins. Josh has 99 of the 36-cent coins (and no other coins). He is allowed to exchange a 36 cent coin for 6 coins of 6 cents, and to exchange a 6 cent coin for 6 coins of 1 cent. Is it possible that after several exchanges Josh will have 500 coins?

p3. Find all solutions $a, b, c, d, e, f, g, h, i$ if these letters represent distinct digits and the following multiplication is correct:

$$\begin{array}{ccccc} & & a & b & c \\ \text{x} & & & d & e \\ \hline & f & a & c & c \\ + & g & h & i & \\ \hline f & f & f & c & c \end{array}$$

p4. Pinocchio rode a bicycle for 3.5 hours. During every 1-hour period he went exactly 5 km. Is it true that his average speed for the trip was 5 km/h? Explain your reasoning.

p5. Let a, b, c be odd integers. Prove that the equation $ax^2 + bx + c = 0$ cannot have a rational solution.

PS. You should use hide for answers. Collected here (https://artofproblemsolving.com/community/c5h2760506p24143309).

2008 **p1.** A square is tiled by smaller squares as shown in the figure. Find the area of the black square in the middle if the perimeter of the square $ABCD$ is 14 cm.
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png

p2. If a, b, and c are numbers so that $a + b + c = 0$ and $a^2 + b^2 + c^2 = 1$. Compute $a^4 + b^4 + c^4$.

p3. A given fraction $\frac{a}{b}$ (a, b are positive integers, $a \neq b$) is transformed by the following rule: first, 1 is added to both the numerator and the denominator, and then the numerator and the denominator of the new fraction are each divided by their greatest common divisor (in other words, the new fraction is put in simplest form). Then the same transformation is applied again and again. Show that after some number of steps the denominator and the numerator differ exactly by 1.

p4. A goat uses horns to make the holes in a new 30×60 cm large towel. Each time it makes two new holes. Show that after the goat repeats this 61 times the towel will have at least two holes whose distance apart is less than 6 cm.

p5. You are given 555 weights weighing 1 g, 2 g, 3 g, ... , 555 g. Divide these weights into three groups whose total weights are equal.

p6. Draw on the regular 8×8 chessboard a circle of the maximal possible radius that intersects only black squares (and does not cross white squares). Explain why no larger circle can satisfy the condition.

PS. You should use hide for answers. Collected here (`https://artofproblemsolving.com/community/c5h2760506p24143309`).

2009 **p1.** Compute the sum of sharp angles at all five nodes of the star below.

(figure missing (`http://www.math.msu.edu/~mshapiro/NewOlympiad/Olymp2009/10_12_2009.pdf`))

p2. Arrange the integers from 1 to 15 in a row so that the sum of any two consecutive numbers is a perfect square. In how many ways this can be done?

p3. Prove that if p and q are prime numbers which are greater than 3 then $p^2 - q^2$ is divisible by 24.

p4. A city in a country is called Large Northern if comparing to any other city of the country it is either larger or farther to the North (or both). Similarly, a city is called Small Southern. We know that in the country all cities are Large Northern city. Show that all the cities in this country are simultaneously Small Southern.

p5. You have four tall and thin glasses of cylindrical form. Place on the flat table these four glasses in such a way that all distances between any pair of centers of the glasses' bottoms are equal.

PS. You should use hide for answers. Collected here (`https://artofproblemsolving.com/community/c5h2760506p24143309`).

2010 **p1.** Find all solutions a, b, c, d, e, f, g if it is known that they represent distinct digits and satisfy the following:

			a	b	c	d
x					a	b
		c	d	b	d	b
+	c	e	b	f	b	
	c	g	a	e	g	b

p2. 5 numbers are placed on the circle. It is known that the sum of any two neighboring numbers is not divisible by 3 and the sum of any three consecutive numbers is not divisible by 3. How many numbers on the circle are divisible by 3?

p3. n teams played in a volleyball tournament. Each team played precisely one game with all other teams. If x_j is the number of victories and y_j is the number of losses of the jth team, show that

$$\sum_{j=1}^{n} x_j^2 = \sum_{j=1}^{n} y_j^2$$

p4. Three cars participated in the car race: a Ford $[F]$, a Toyota $[T]$, and a Honda $[H]$. They began the race with F first, then T, and H last. During the race, F was passed a total of 3 times, T was passed 5 times, and H was passed 8 times. In what order did the cars finish?

p5. The side of the square is 4 cm. Find the sum of the areas of the six half-disks shown on the picture.
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PS. You should use hide for answers. Collected here (https://artofproblemsolving.com/community/c5h2760506p24143309).

2012 **p1.** A triangle ABC is drawn in the plane. A point D is chosen inside the triangle. Show that the sum of distances $AD + BD + CD$ is less than the perimeter of the triangle.

p2. In a triangle ABC the bisector of the angle C intersects the side AB at M, and the bisector of the angle A intersects CM at the point T. Suppose that the segments CM and AT divided the triangle ABC into three isosceles triangles. Find the angles of the triangle ABC.

p3. You are given 100 weights of masses $1, 2, 3, ..., 99, 100$. Can one distribute them into 10 piles having the following property: the heavier the pile, the fewer weights it contains?

p4. Each cell of a 10×10 table contains a number. In each line the greatest number (or one of the largest, if more than one) is underscored, and in each column the smallest (or one of the smallest) is also underscored. It turned out that all of the underscored numbers are underscored exactly twice. Prove that all numbers stored in the table are equal to each other.

p5. Two stores have warehouses in which wheat is stored. There are 16 more tons of wheat in the first warehouse than in the second. Every night exactly at midnight the owner of each store steals from his rival, taking a quarter of the wheat in his rival's warehouse and dragging it to his own. After 10 days, the thieves are caught. Which warehouse has more wheat at this point and by how much?

PS. You should use hide for answers. Collected here (`https://artofproblemsolving.com/community/c5h2760506p24143309`).

2013 **p1.** A function f defined on the set of positive numbers satisfies the equality

$$f(xy) = f(x) + f(y), x, y > 0.$$

Find $f(2007)$ if $f\left(\frac{1}{2007}\right) = 1$.

p2. The plane is painted in two colors. Show that there is an isosceles right triangle with all vertices of the same color.

p3. Show that the number of ways to cut a $2n \times 2n$ square into 1×2 dominoes is divisible by 2.

p4. Two mirrors form an angle. A beam of light falls on one mirror. Prove that the beam is reflected only finitely many times (even if the angle between mirrors is very small).

p5. A sequence is given by the recurrence relation $a_{n+1} = (s(a_n))^2 + 1$, where $s(x)$ is the sum of the digits of the positive integer x. Prove that starting from some moment the sequence is periodic.

PS. You should use hide for answers. Collected here (`https://artofproblemsolving.com/community/c5h2760506p24143309`).

2014 **p1.** The length of the side AB of the trapezoid with bases AD and BC is equal to the sum of lengths $|AD| + |BC|$. Prove that bisectors of angles A and B do intersect at a point of the side CD.

p2. Polynomials $P(x) = x^4 + ax^3 + bx^2 + cx + 1$ and $Q(x) = x^4 + cx^3 + bx^2 + ax + 1$ have two common roots. Find these common roots of both polynomials.

p3. A girl has a box with 1000 candies. Outside the box there is an infinite number of chocolates and muffins. A girl may replace: • two candies in the box with one chocolate bar, • two muffins in the box with one chocolate bar, • two chocolate bars in the box with one candy and one muffin, • one candy and one chocolate bar in the box with one muffin, • one muffin and one chocolate bar in the box with one candy.
Is it possible that after some time it remains only one object in the box?

p4. There are 9 straight lines drawn in the plane. Some of them are parallel some of them intersect each other. No three lines do intersect at one point. Is it possible to have exactly 17 intersection points?

p5. It is known that x is a real number such that $x + \frac{1}{x}$ is an integer. Prove that $x^n + \frac{1}{x^n}$ is an integer for any positive integer n.

PS. You should use hide for answers. Collected here (https://artofproblemsolving.com/community/c5h2760506p24143309).

2015 **p1.** What is the maximal number of pieces of two shapes, https://cdn.artofproblemsolving.com/attachments/a/5/6c567cf6a04b0aa9e998dbae3803b6eeb24a35.png and https://cdn.artofproblemsolving.com/attachments/8/a/7a7754d0f2517c93c5bb931fb7b5ae8f5e3217.png, that can be used to tile a 7×7 square?

p2. Six shooters participate in a shooting competition. Every participant has 5 shots. Each shot adds from 1 to 10 points to shooter's score. Every person can score totally for all five shots from 5 to 50 points. Each participant gets 7 points for at least one of his shots. The scores of all participants are different. We enumerate the shooters 1 to 6 according to their scores, the person with maximal score obtains number 1, the next one obtains number 2, the person with minimal score obtains number 6. What score does obtain the participant number 3? The total number of all obtained points is 264.

p2. There are exactly n students in a high school. Girls send messages to boys. The first girl sent messages to 5 boys, the second to 7 boys, the third to 6 boys, the fourth to 8 boys, the fifth to 7 boys, the sixth to 9 boys, the seventh to 8, etc. The last girl sent messages to all the boys. Prove that n is divisible by 3.

p4. In what minimal number of triangles can one cut a 25×12 rectangle in such a way that one can tile by these triangles a 20×15 rectangle.

p5. There are 2014 stones in a pile. Two players play the following game. First, player A takes some number of stones (from 1 to 30) from the pile, then player B takes 1 or 2 stones, then player A takes 2 or 3 stones, then player B takes 3 or 4 stones, then player A takes 4 or 5 stones, etc. The player who gets the last stone is the winner. If no player gets the last stone (there is at least one stone in the pile but the next move is not allowed) then the game results in a draw. Who wins the game using the right strategy?

PS. You should use hide for answers. Collected here (https://artofproblemsolving.com/community/c5h2760506p24143309).

2017 **p1.** In the group of five people any subgroup of three persons contains at least two friends. Is it possible to divide these five people into two subgroups such that all members of any subgroup are friends?

p2. Coefficients a, b, c in expression $ax^2 + bx + c$ are such that $b - c > a$ and $a \neq 0$. Is it true that equation $ax^2 + bx + c = 0$ always has two distinct real roots?

p3. Point D is a midpoint of the median AF of triangle ABC. Line CD intersects AB at point E. Distances $|BD| = |BF|$. Show that $|AE| = |DE|$.

p4. Real numbers a, b satisfy inequality $a + b^5 > ab^5 + 1$. Show that $a + b^7 > ba^7 + 1$.

p5. A positive number was rounded up to the integer and got the number that is bigger than the original one by 28%. Find the original number (find all solutions).

p6. Divide a 5×5 square along the sides of the cells into 8 parts in such a way that all parts are different.

PS. You should use hide for answers. Collected here (https://artofproblemsolving.com/community/c5h2760506p24143309).

2018 **p1.** Twenty five horses participate in a competition. The competition consists of seven runs, five horse compete in each run. Each horse shows the same result in any run it takes part. No two horses will give the same result. After each run you can decide what horses participate in the next run. Could you determine the three fastest horses? (You don't have stopwatch. You can only remember the order of the horses.)

p2. Prove that the equation $x^6 - 143x^5 - 917x^4 + 51x^3 + 77x^2 + 291x + 1575 = 0$ does not have solutions in integer numbers.

p3. Show how we can cut the figure shown in the picture into two parts for us to be able to assemble a square out of these two parts. Show how we can assemble a square.
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png

p4. The city of Vyatka in Russia produces local drink, called "Vyatka Cola". "Vyatka Cola" is sold in 1, $3/4$, and $1/2$-gallon bottles. Ivan and John bought 4 gallons of "Vyatka Cola". Can we say for sure, that they can split the Cola evenly between them without opening the bottles?

p5. Positive numbers a, b and c satisfy the condition $a+bc = (a+b)(a+c)$. Prove that $b+ac = (b+a)(b+c)$.

PS. You should use hide for answers. Collected here (https://artofproblemsolving.com/community/c5h2760506p24143309).

2019 **p1.** In triangle ABC, the median BM is drawn. The length $|BM| = |AB|/2$. The angle $\angle ABM = 50^o$. Find the angle $\angle ABC$.

p2. Is there a positive integer n which is divisible by each of $1, 2, 3, ..., 2018$ except for two numbers whose difference is7?

p3. Twenty numbers are placed around the circle in such a way that any number is the average of its two neighbors. Prove that all of the numbers are equal.

p4. A finite number of frogs occupy distinct integer points on the real line. At each turn, a single frog jumps by 1 to the right so that all frogs again occupy distinct points. For some initial configuration, the frogs can make n moves in m ways. Prove that if they jump by 1 to the left (instead of right) then the number of ways to make n moves is also m.

p5. A square box of chocolates is divided into 49 equal square cells, each containing either dark or white chocolate. At each move Alex eats two chocolates of the same kind if they are in adjacent cells (sharing a side or a vertex). What is the maximal number of chocolates Alex can eat regardless of distribution of chocolates in the box?

PS. You should use hide for answers. Collected here (https://artofproblemsolving.com/community/c5h2760506p24143309).

2022 **p1.** Consider a triangular grid: nodes of the grid are painted black and white. At a single step you are allowed to change colors of all nodes situated on any straight line (with the slope 0^o ,60^o, or 120^o) going through the nodes of the grid. Can you transform the combination in the left picture into the one in the right picture in a finite number of steps?
`https://cdn.artofproblemsolving.com/attachments/3/a/957b199149269ce1d0f66b91a1ac0737cf3f8`
`png`

p2. Find x satisfying $\sqrt{x\sqrt{x\sqrt{x...}}} = \sqrt{2022}$ where it is an infinite expression on the left side.

p3. 179 glasses are placed upside down on a table. You are allowed to do the following moves. An integer number k is fixed. In one move you are allowed to turn any k glasses .
(a) Is it possible in a finite number of moves to turn all 179 glasses into "bottom-down" positions if $k = 3$?
(b) Is it possible to do it if $k = 4$?

p4. An interval of length 1 is drawn on a paper. Using a compass and a simple ruler construct an interval of length $\sqrt{93}$.

p5. Show that $5^{2n+1} + 3^{n+2}2^{n-1}$ is divisible by 19 for any positive integer n.

p6. Solve the system

$$\begin{cases} \dfrac{xy}{x+y} = 1 - z \\ \dfrac{yz}{y+z} = 2 - x \\ \dfrac{xz}{x+z} = 2 - y \end{cases}$$

PS. You should use hide for answers. Collected here (`https://artofproblemsolving.com/community/c5h2760506p24143309`).

2023 **p1.** There are 16 students in a class. Each month the teacher divides the class into two groups. What is the minimum number of months that must pass for any two students to be in different groups in at least one of the months?

p2. Find all functions $f(x)$ defined for all real x that satisfy the equation $2f(x) + f(1 - x) = x^2$.

p3. Arrange the digits from 1 to 9 in a row (each digit only once) so that every two consecutive digits form a two-digit number that is divisible by 7 or 13.

p4. Prove that $\cos 1^o$ is irrational.

p5. Consider $2n$ distinct positive Integers $a_1, a_2, ..., a_{2n}$ not exceeding n^2 ($n > 2$). Prove that some three of the differences $a_i - a_j$ are equal .

PS. You should use hide for answers. Collected here (https://artofproblemsolving.com/community/c5h2760506p24143309).
