USAJMO 2022

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– **Day 1** March 22nd

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1 For which positive integers m does there exist an infinite arithmetic sequence of integers $a_1, a_2, \ldots$ and an infinite geometric sequence of integers $g_1, g_2, \ldots$ satisfying the following properties?

- $a_n - g_n$ is divisible by m for all integers $n \geq 1$;
- $a_2 - a_1$ is not divisible by m.

Holden Mui

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2 Let a and b be positive integers. The cells of an $(a+b+1) \times (a+b+1)$ grid are colored amber and bronze such that there are at least $a^2 + ab - b$ amber cells and at least $b^2 + ab - a$ bronze cells. Prove that it is possible to choose a amber cells and b bronze cells such that no two of the $a+b$ chosen cells lie in the same row or column.

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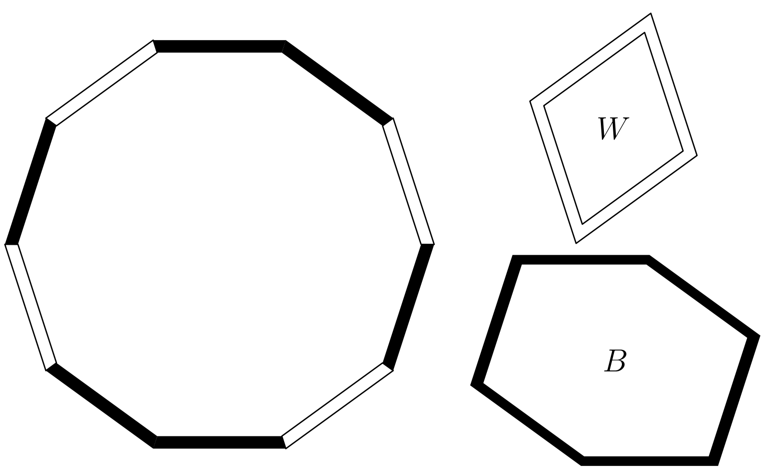
3 Let $b \geq 2$ and $w \geq 2$ be fixed integers, and $n = b + w$. Given are $2b$ identical black rods and $2w$ identical white rods, each of side length 1.

We assemble a regular $2n-$gon using these rods so that parallel sides are the same color. Then, a convex $2b$-gon B is formed by translating the black rods, and a convex $2w$-gon W is formed by translating the white rods. An example of one way of doing the assembly when $b = 3$ and $w = 2$ is shown below, as well as the resulting polygons B and W.



Prove that the difference of the areas of B and W depends only on the numbers b and w, and not on how the $2n$-gon was assembled.

Proposed by Ankan Bhattacharya

– **Day 2** March 23rd

4 Let $ABCD$ be a rhombus, and let K and L be points such that K lies inside the rhombus, L lies outside the rhombus, and $KA = KB = LC = LD$. Prove that there exist points X and Y on lines AC and BD such that $KXLY$ is also a rhombus.

Proposed by Ankan Bhattacharya

5 Find all pairs of primes (p, q) for which $p - q$ and $pq - q$ are both perfect squares.

6 Let a_0, b_0, c_0 be complex numbers, and define

$$a_{n+1} = a_n^2 + 2b_n c_n$$
$$b_{n+1} = b_n^2 + 2c_n a_n$$
$$c_{n+1} = c_n^2 + 2a_n b_n$$

for all nonnegative integers n.

Suppose that $\max\{|a_n|, |b_n|, |c_n|\} \leq 2022$ for all n. Prove that

$$|a_0|^2 + |b_0|^2 + |c_0|^2 \leq 1.$$