USAMO 2022

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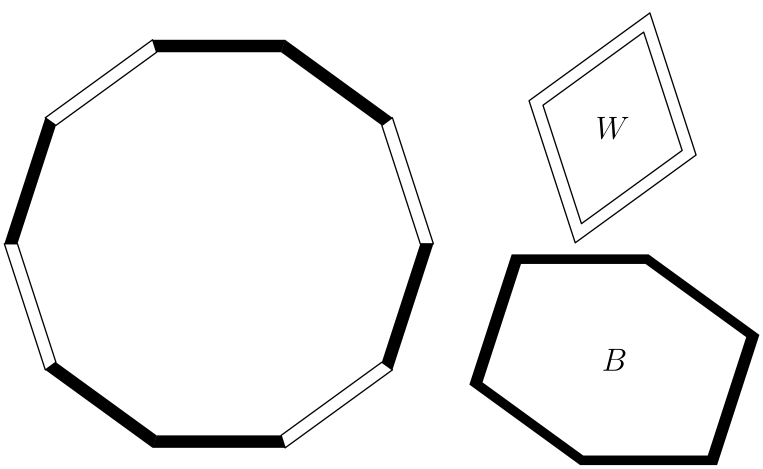
by inventivedant, AwesomeYRY, DottedCaculator, rrusczyk

– **Day 1** March 22

1 Let a and b be positive integers. The cells of an $(a+b+1)\times(a+b+1)$ grid are colored amber and bronze such that there are at least a^2+ab-b amber cells and at least b^2+ab-a bronze cells. Prove that it is possible to choose a amber cells and b bronze cells such that no two of the $a+b$ chosen cells lie in the same row or column.

2 Let $b\geq 2$ and $w\geq 2$ be fixed integers, and $n=b+w$. Given are $2b$ identical black rods and $2w$ identical white rods, each of side length 1.

We assemble a regular $2n-$gon using these rods so that parallel sides are the same color. Then, a convex $2b$-gon B is formed by translating the black rods, and a convex $2w$-gon W is formed by translating the white rods. An example of one way of doing the assembly when $b=3$ and $w=2$ is shown below, as well as the resulting polygons B and W.



Prove that the difference of the areas of B and W depends only on the numbers b and w, and not on how the $2n$-gon was assembled.

Proposed by Ankan Bhattacharya

3 Let $\mathbb{R}_{>0}$ be the set of all positive real numbers. Find all functions $f:\mathbb{R}_{>0}\to\mathbb{R}_{>0}$ such that for all $x,y\in\mathbb{R}_{>0}$ we have

$$f(x)=f(f(f(x))+y)+f(xf(y))f(x+y).$$

–	**Day 2** March 23
4	Find all pairs of primes (p, q) for which $p - q$ and $pq - q$ are both perfect squares.
5	A function $f : \mathbb{R} \to \mathbb{R}$ is *essentially increasing* if $f(s) \leq f(t)$ holds whenever $s \leq t$ are real numbers such that $f(s) \neq 0$ and $f(t) \neq 0$. Find the smallest integer k such that for any 2022 real numbers $x_1, x_2, \ldots, x_{2022}$, there exist k essentially increasing functions $f_1, \ldots, f_k$ such that $$f_1(n) + f_2(n) + \cdots + f_k(n) = x_n \qquad \text{for every } n = 1, 2, \ldots 2022.$$
6	There are 2022 users on a social network called Mathbook, and some of them are Mathbook-friends. (On Mathbook, friendship is always mutual and permanent.) Starting now, Mathbook will only allow a new friendship to be formed between two users if they have *at least two* friends in common. What is the minimum number of friendships that must already exist so that every user could eventually become friends with every other user?
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