Caltech Harvey Mudd Math Competition from Fall 2013

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by parmenides51

–	Team Round
1	In how many ways can you rearrange the letters of 'Alejandro' such that it contains one of the words 'ned' or 'den'?
2	Two circles of radii 7 and 17 have a distance of 25 between their centers. What is the difference between the lengths of their common internal and external tangents (positive difference)?
3	Let p_n be the product of the nth roots of 1. For integral $x > 4$, let $f(x) = p_1 - p_2 + p_3 - p_4 + ... + (-1)^{x+1}p_x$. What is $f(2010)$?
4	The numbers 25 and 76 have the property that when squared in base 10, their squares also end in the same two digits. A positive integer that has at most 3 digits when expressed in base 21 and also has the property that its base 21 square ends in the same 3 digits is called amazing. Find the sum of all amazing numbers. Express your answer in base 21.
5	Compute the number of lattice points bounded by the quadrilateral formed by the points $(0, 0)$, $(0, 140)$, $(140, 0)$, and $(100, 100)$ (including the quadrilateral itself). A lattice point on the xy -plane is a point (x, y), where both x and y are integers.
6	Let $a_1 < a_2 < a_3 < ... < a_n < ...$ be positive integers such that, for $n = 1, 2, 3, ...,$ $$a_{2n} = a_n + n.$$ Given that if a_n is prime, then n is also, find a_{2014}.
7	The points $(0, 0)$, $(a, 5)$, and $(b, 11)$ are the vertices of an equilateral triangle. Find ab.
8	Two kids A and B play a game as follows: from a box containing n marbles ($n > 1$), they alternately take some marbles for themselves, such that: 1. A goes first. 2. The number of marbles taken by A in his first turn, denoted by k, must be between 1 and $n-1$, inclusive. 3. The number of marbles taken in a turn by any player must be between 1 and k, inclusive. The winner is the one who takes the last marble. Determine all natural numbers n for which A has a winning strategy

9 A 7×7 grid of unit-length squares is given. Twenty-four 1×2 dominoes are placed in the grid, each covering two whole squares and in total leaving one empty space. It is allowed to take a domino adjacent to the empty square and slide it lengthwise to fill the whole square, leaving a new one empty and resulting in a different configuration of dominoes. Given an initial configuration of dominoes for which the maximum possible number of distinct configurations can be reached through any number of slides, compute the maximum number of distinct configurations.

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10 Compute the lowest positive integer k such that none of the numbers in the sequence

$$\{1, 1+k, 1+k+k^2, 1+k+k^2+k^3, ...\}$$

are prime.

– Tiebreaker Round

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1 In the diagram below, point A lies on the circle centered at O. AB is tangent to circle O with $\overline{AB}=6$. Point C is $\frac{2\pi}{3}$ radians away from point A on the circle, with BC intersecting circle O at point D. The length of BD is 3. Compute the radius of the circle.

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png

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2 Suppose the roots of

$$x^4-3x^2+6x-12=1$$

are α, β, γ , and δ. What is the value of

$$\frac{\alpha+\beta+\gamma}{\delta^2}+\frac{\alpha+\delta+\gamma}{\beta^2}+\frac{\alpha+\beta+\delta}{\gamma^2}+\frac{\delta+\beta+\gamma}{\alpha^2}?$$

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3 Bill plays a game in which he rolls two fair standard six-sided dice with sides labeled one through six. He wins if the number on one of the dice is three times the number on the other die. If Bill plays this game three times, compute the probability that he wins at least once.

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4 Let

$$A=\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{9},$$
$$B=\frac{1}{2\cdot3}+\frac{1}{2\cdot5}+\frac{1}{2\cdot9}+\frac{1}{3\cdot5}+\frac{1}{3\cdot9}+\frac{1}{5\cdot9},$$
$$C=\frac{1}{2\cdot3\cdot5}+\frac{1}{2\cdot3\cdot9}+\frac{1}{2\cdot5\cdot9}+\frac{1}{3\cdot5\cdot9}.$$

Compute the value of $A+B+C$.

– Mixer Round

Mixer Part 1

p1. Two kids A and B play a game as follows: From a box containing n marbles ($n > 1$), they alternately take some marbles for themselves, such that:
1. A goes first.
2. The number of marbles taken by A in his first turn, denoted by k, must be between 1 and n, inclusive.
3. The number of marbles taken in a turn by any player must be between 1 and k, inclusive.
The winner is the one who takes the last marble. What is the sum of all n for which B has a winning strategy?

p2. How many ways can your rearrange the letters of "Alejandro" such that it contains exactly one pair of adjacent vowels?

p3. Assuming real values for p, q, r, and s, the equation

$$x^4 + px^3 + qx^2 + rx + s$$

has four non-real roots. The sum of two of these roots is $q + 6i$, and the product of the other two roots is $3 - 4i$. Find the smallest value of q.

p4. Lisa has a 3D box that is 48 units long, 140 units high, and 126 units wide. She shines a laser beam into the box through one of the corners, at a 45^o angle with respect to all of the sides of the box. Whenever the laser beam hits a side of the box, it is reflected perfectly, again at a 45^o angle. Compute the distance the laser beam travels until it hits one of the eight corners of the box.

Part 2

p5. How many ways can you divide a heptagon into five non-overlapping triangles such that the vertices of the triangles are vertices of the heptagon?

p6. Let a be the greatest root of $y = x^3 + 7x^2 - 14x - 48$. Let b be the number of ways to pick a group of a people out of a collection of a^2 people. Find $\frac{b}{2}$.

p7. Consider the equation

$$1 - \frac{1}{d} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c},$$

with a, b, c, and d being positive integers. What is the largest value for d?

p8. The number of non-negative integers $x_1, x_2, ..., x_{12}$ such that

$$x_1 + x_2 + ... + x_{12} \le 17$$

can be expressed in the form $\binom{a}{b}$, where $2b \le a$. Find $a + b$.

Part 3

p9. In the diagram below, AB is tangent to circle O. Given that $AC = 15$, $AB = 27/2$, and $BD = 243/34$, compute the area of $\triangle ABC$.
`https://cdn.artofproblemsolving.com/attachments/b/f/b403e5e188916ac4fb1b0ba74adb7f1e50e86`
`png`

p10. If

$$\left[2^{\log x}\right]^{[x^{\log 2}]^{[2^{\log x}]...}} = 2,$$

where $\log x$ is the base-10 logarithm of x, then it follows that $x = \sqrt{n}$. Compute n^2.

p11.

p12. Find n in the equation

$$133^5 + 110^5 + 84^5 + 27^5 = n^5,$$

where n is an integer less than 170.

Part 4

p13. Let x be the answer to number 14, and z be the answer to number 16. Define $f(n)$ as the number of distinct two-digit integers that can be formed from digits in n. For example, $f(15) = 4$ because the integers 11, 15, 51, 55 can be formed from digits of 15. Let w be such that $f(3xz - w) = w$. Find w.

p14. Let w be the answer to number 13 and z be the answer to number 16. Let x be such that the coefficient of a^xb^x in $(a + b)^{2x}$ is $5z^2 + 2w - 1$. Find x.

p15. Let w be the answer to number 13, x be the answer to number 14, and z be the answer to number 16. Let A, B, C, D be points on a circle, in that order, such that $\overline{AD}$ is a diameter of the circle. Let E be the intersection of $\overleftrightarrow{AB}$ and $\overleftrightarrow{DC}$, let F be the intersection of $\overleftrightarrow{AC}$ and $\overleftrightarrow{BD}$, and let

G be the intersection of $\overleftrightarrow{EF}$ and $\overleftrightarrow{AD}$. Now, let $AE = 3x$, $ED = w^2 - w + 1$, and $AD = 2z$. If $FG = y$, find y.

p16. Let w be the answer to number 13, and x be the answer to number 16. Let z be the number of integers n in the set $S = \{w, w+1, ..., 16x-1, 16x\}$ such that $n^2 + n^3$ is a perfect square. Find z.

PS. You had better use hide for answers. Collected here (`https://artofproblemsolving.com/community/c5h2760506p24143309`).

– Individual Round

Individual p1. Compute

$$\sqrt{(\sqrt{63}+\sqrt{112}+\sqrt{175})(-\sqrt{63}+\sqrt{112}+\sqrt{175})(\sqrt{63}-\sqrt{112}+\sqrt{175})(\sqrt{63}+\sqrt{112}-\sqrt{175})}$$

p2. Consider the set $S = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. How many distinct 3-element subsets are there such that the sum of the elements in each subset is divisible by 3?

p3. Let a^2 and b^2 be two integers. Consider the triangle with one vertex at the origin, and the other two at the intersections of the circle $x^2 + y^2 = a^2 + b^2$ with the graph $ay = b|x|$. If the area of the triangle is numerically equal to the radius of the circle, what is this area?

p4. Suppose $f(x) = x^3 + x - 1$ has roots a, b and c. What is

$$\frac{a^3}{1-a} + \frac{b^3}{1-b} + \frac{c^3}{1-c}?$$

p5. Lisa has a $2D$ rectangular box that is 48 units long and 126 units wide. She shines a laser beam into the box through one of the corners such that the beam is at a 45^o angle with respect to the sides of the box. Whenever the laser beam hits a side of the box, it is reflected perfectly, again at a 45^o angle. Compute the distance the laser beam travels until it hits one of the four corners of the box.

p6. How many ways can we form a group with an odd number of members (plural) from 99 people total?
Express your answer in the form $a^b + c$, where a, b, and c are integers, and a is prime.

p7. Let

$$S = \log_2 9 \log_3 16 \log_4 25 ... \log_{999} 1000000.$$

Compute the greatest integer less than or equal to $\log_2 S$.

p8. A prison, housing exactly four hundred prisoners in four hundred cells numbered 1-400, has a really messed-up warden. One night, when all the prisoners are asleep and all of their doors are locked, the warden toggles the locks on all of their doors (that is, if the door is locked, he unlocks the door, and if the door is unlocked, he locks it again), starting at door 1 and ending at door 400. The warden then toggles the lock on every other door starting at door 2 (2, 4, 6, etc). After he has toggled the lock on every other door, the warden then toggles every third door (doors $3, 6, 9$, etc.), then every fourth door, etc., finishing by toggling every 400th door (consisting of only the 400th door). He then collapses in exhaustion.
Compute the number of prisoners who go free (that is, the number of unlocked doors) when they wake up the next morning.

p9. Let A and B be fixed points on a 2-dimensional plane with distance $AB = 1$. An ant walks on a straight line from point A to some point C on the same plane and finds that the distance from itself to B always decreases at any time during this walk. Compute the area of the locus of points where point C could possibly be located.

p10. A robot starts in the bottom left corner of a 4×4 grid of squares. How many ways can it travel to each square exactly once and then return to its start if it is only allowed to move to an adjacent (not diagonal) square at each step?

p11. Assuming real values for p, q, r, and s, the equation

$$x^4 + px^3 + qx^2 + rx + s$$

has four non-real roots. The sum of two of these roots is $4 + 7i$, and the product of the other two roots is $3 - 4i$. Find q.

p12. A cube is inscribed in a right circular cone such that one face of the cube lies on the base of the cone. If the ratio of the height of the cone to the radius of the cone is $2 : 1$, what fraction of the cone's volume does the cube take up? Express your answer in simplest radical form.

p13. Let

$$y = \cfrac{1}{1 + \cfrac{1}{9 + \cfrac{1}{5 + \cfrac{1}{9 + \cfrac{1}{5 + ...}}}}}$$

If y can be represented as $\frac{a\sqrt{b}+c}{d}$, where b is not divisible by the square of any prime, and the greatest common divisor of a and d is 1, find the sum $a+b+c+d$.

p14. Alice wants to paint each face of an octahedron either red or blue. She can paint any number of faces a particular color, including zero. Compute the number of ways in which she can do this. Two ways of painting the octahedron are considered the same if you can rotate the octahedron to get from one to the other.

p15. Find n in the equation

$$133^5 + 110^5 + 84^5 + 27^5 = n^5,$$

where n is an integer less than 170.

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