Caltech Harvey Mudd Math Competition from Fall 2014

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by parmenides51

– Team Round

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1 Suppose we have a hexagonal grid in the shape of a hexagon of side length 4 as shown at left. Define a "chunk" to be four tiles, two of which are adjacent to the other three, and the other two of which are adjacent to just two of the others. The three possible rotations of these are shown at right.
`https://cdn.artofproblemsolving.com/attachments/a/7/147d8aa2c149918ab855db1e945d389433446`
`png`
In how many ways can we choose a chunk from the grid?

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2 Consider two overlapping regular tetrahedrons of side length 2 in space. They are centered at the same point, and the second one is oriented so that the lines from its center to its vertices are perpendicular to the faces of the first tetrahedron. What is the volume encompassed by the combined solid?

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3 Suppose that in a group of 6 people, if A is friends with B, then B is friends with A. If each of the 6 people draws a graph of the friendships between the other 5 people, we get these 6 graphs, where edges represent
friendships and points represent people.
`https://cdn.artofproblemsolving.com/attachments/5/5/7265067f585e3dfe77ba94ac6261b4462cd01`
`png`
If Sue drew the first graph, how many friends does she have?

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4 Let $b_1 = 1$ and $b_{n+1} = 1 + \frac{1}{n(n+1)b_1b_2...b_n}$ for $n \geq 1$. Find b_12.

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5 A teacher gives a multiple choice test to 15 students and that each student answered each question. Each question had 5 choices, but remarkably, no pair of students had more than 2 answers in common. What is the maximum number of questions that could have been on the quiz?

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6 Suppose the transformation T acts on points in the plane like this:

$$T(x, y) = \left(\frac{x}{x^2 + y^2}, \frac{-y}{x^2 + y^2}\right).$$

Determine the area enclosed by the set of points of the form $T(x, y)$, where (x, y) is a point on the edge of a length-2 square centered at the origin with sides parallel to the axes.

7 Let

$$P(x)=\sum_{k=1}^{n}(x^{3^k}+x^{-3^k}-1), Q(x)=\sum_{k=1}^{n}(x^{3^k}+x^{-3^k}+1).$$

Given that

$$P(x)Q(x)=\sum_{k=-2\cdot 3^n}^{2\cdot 3^n} a_k x^k,$$

Compute $\sum_{k=0}^{3^n} a_k$ in terms of n.

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8 What's the greatest pyramid volume one can form using edges of length $2, 3, 3, 4, 5, 5$, respectively?

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9 There is a long-standing conjecture that there is no number with $2n+1$ instances in Pascal's triangle for $n \geq 2$. Assuming this is true, for how many $n \leq 100,000$ are there exactly 3 instances of n in Pascal's triangle?

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10 Consider a grid of all lattice points (m, n) with m, n between 1 and 125. There exists a "path" between two lattice points (m_1, n_1) and (m_2, n_2) on the grid if $m_1n_1 = m_2n_2$ or if $m_1/n_1 = m_2/n_2$. For how many lattice points (m, n) on the grid is there a sequence of paths that goes from $(1, 1)$ to (m, n)?

– Tiebreaker Round

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1 For $a_1, ..., a_5 \in R$,

$$\frac{a_1}{k^2+1}+...+\frac{a_5}{k^2+5}=\frac{1}{k^2}$$

for all $k \in \{2, 3, 4, 5, 6\}$. Calculate

$$\frac{a_1}{2}+...+\frac{a_5}{6}.$$

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2 A matrix $\begin{bmatrix} x & y \\ z & w \end{bmatrix}$ has square root $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ if

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^2=\begin{bmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{bmatrix}=\begin{bmatrix} x & y \\ z & w \end{bmatrix}$$

Determine how many square roots the matrix $\begin{bmatrix} 2 & 2 \\ 3 & 4 \end{bmatrix}$ has (complex coefficients are allowed).

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3 Two players play a game on a pile of n beans. On each player's turn, they may take exactly 1, 4, or 7 beans from the pile. One player goes first, and then the players alternate until somebody wins. A player wins when they take the last bean from the pile. For how many n between 2014 and 2050 (inclusive) does the second player win?

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4 If $f(i, j, k) = f(i-1, j+k, 2i-1)$ and $f(0, j, k) = j + k$, evaluate $f(n, 0, 0)$.

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5 Determine the value of

$$\prod_{n=1}^{\infty} 3^{n/3^n} = \sqrt[3]{3}\ \sqrt[3^2]{3^2}\ \sqrt[3^3]{3^3}...$$

– Mixer Round

Mixer Fermi Questions

p1. What is $\sin(1000)$? (note: that's 1000 radians, not degrees)

p2. In liters, what is the volume of 10 million US dollars' worth of gold?

p3. How many trees are there on Earth?

p4. How many prime numbers are there between 10^8 and 10^9?

p5. What is the total amount of time spent by humans in spaceflight?

p6. What is the global domestic product (total monetary value of all goods and services produced in a country's borders in a year) of Bangladesh in US dollars?

p7. How much time does the average American spend eating during their lifetime, in hours?

p8. How many CHMMC-related emails did the directors receive or send in the last month?

Suspiciously Familiar. . .

p9. Suppose a farmer learns that he will die at the end of the year (day 365, where today is day 0) and that he has 100 sheep. He decides to sell all his sheep on one day, and that his utility is given by ab where a is the money he makes by selling the sheep (which always have a fixed price) and b is the number of days he has left to enjoy the profit; i.e., $365 - k$ where k is the day number. If every day his sheep breed and multiply their numbers by $(421 + b)/421$ (yes, there are small, fractional sheep), on which day should he sell out?

p10. Suppose in your sock drawer of 14 socks there are 5 different colors and 3 different lengths present. One day, you decide you want to wear two socks that have either different colors or different lengths but not both. Given only this information, what is the maximum number of choices you might have?

I'm So Meta Even This Acronym

p11. Let $\frac{s}{t}$ be the answer of problem 13, written in lowest terms. Let $\frac{p}{q}$ be the answer of problem 12, written in lowest terms.
If player 1 wins in problem 11, let $n = q$. Otherwise, let $n = p$.
Two players play a game on a connected graph with n vertices and t edges. On each player's turn, they remove one edge of the graph, and lose if this causes the graph to become disconnected. Which player (first or second) wins?

p12. Let $\frac{s}{t}$ be the answer of problem 13, written in lowest terms.
If player 1 wins in problem 11, let $n = t$. Otherwise, let $n = s$.
Find the maximum value of
$$\frac{x^n}{1+\frac{1}{2}x+\frac{1}{4}x^2+...+\frac{1}{2^{2n}}x^{2n}}$$
for $x > 0$.

p13. Let $\frac{p}{q}$ be the answer of problem 12, written in lowest terms.
Let y be the largest integer such that 2^y divides p.
If player 1 wins in problem 11, let $z = q$. Otherwise, let $z = p$.
Suppose that $a_1 = 1$ and
$$a_{n+1} = a_n - \frac{z}{n+2} + \frac{2z}{n+1} - \frac{z}{n}$$
What is a_y?

PS. You had better use hide for answers. Collected here (https://artofproblemsolving.com/community/c5h2760506p24143309).

– Individual Round

Individual **p1.** In the following 3 by 3 grid, a, b, c are numbers such that the sum of each row is listed at the right and the sum of each column is written below it:

https://cdn.artofproblemsolving.com/attachments/d/9/4f6fd2bc959c25e49add58e6e09a7b7eed9346.png

What is n?

p2. Suppose in your sock drawer of 14 socks there are 5 different colors and 3 different lengths present. One day, you decide you want to wear two socks that have both different colors and different lengths. Given only this information, what is the maximum number of choices you might have?

p3. The population of Arveymuddica is 2014, which is divided into some number of equal groups. During an election, each person votes for one of two candidates, and the person who was voted for by $2/3$ or more of the group wins. When neither candidate gets $2/3$ of the vote, no one wins the group. The person who wins the most groups wins the election. What should the size of the groups be if we want to minimize the minimum total number of votes required to win an election?

p4. A farmer learns that he will die at the end of the year (day 365, where today is day 0) and that he has a number of sheep. He decides that his utility is given by ab where a is the money he makes by selling his sheep (which always have a fixed price) and b is the number of days he has left to enjoy the profit; i.e., $365 - k$ where k is the day. If every day his sheep breed and multiply their numbers by $103/101$ (yes, there are small, fractional sheep), on which day should he sell them all?

p5. Line segments $\overline{AB}$ and $\overline{AC}$ are tangent to a convex arc BC and $\angle BAC = \frac{\pi}{3}$. If $\overline{AB} = \overline{AC} = 3\sqrt{3}$, find the length of arc BC.

p6. Suppose that you start with the number 8 and always have two legal moves: • Square the number • Add one if the number is divisible by 8 or multiply by 4 otherwise
How many sequences of 4 moves are there that return to a multiple of 8?

p7. A robot is shuffling a 9 card deck. Being very well machined, it does every shuffle in exactly the same way: it splits the deck into two piles, one containing the 5 cards from the bottom of the deck and the other with the 4 cards from the top. It then interleaves the cards from the two piles, starting with a card from the bottom of the larger pile at the bottom of the new deck, and then alternating cards from the two piles while maintaining the relative order of each pile. The top card of the new deck will be the top card of the bottom pile. The robot repeats this shuffling procedure a total of n times, and notices that the cards are in the same order as they were when it started shuffling. What is the smallest possible value of n?

p8. A secant line incident to a circle at points A and C intersects the circle's diameter at point B with a 45^o angle. If the length of AB is 1 and the length of BC is 7, then what is the circle's radius?

p9. If a complex number z satisfies $z + 1/z = 1$, then what is $z^{96} + 1/z^{96}$?

p10. Let a, b be two acute angles where $\tan a = 5 \tan b$. Find the maximum possible value of $\sin(a - b)$.

p11. A pyramid, represented by $SABCD$ has parallelogram $ABCD$ as base (A is across from C) and vertex S. Let the midpoint of edge SC be P. Consider plane $AMPN$ where$M$ is on edge SB and N is on edge SD. Find the minimum value r_1 and maximum value r_2 of $\frac{V_1}{V_2}$ where V_1 is the volume of pyramid $SAMPN$ and V_2 is the volume of pyramid $SABCD$. Express your answer as an ordered pair (r_1, r_2).

p12. A 5×5 grid is missing one of its main diagonals. In how many ways can we place 5 pieces on the grid such that no two pieces share a row or column?

p13. There are 20 cities in a country, some of which have highways connecting them. Each highway goes from one city to another, both ways. There is no way to start in a city, drive along the highways of the country such that you travel through each city exactly once, and return to the same city you started in. What is the maximum number of roads this country could have?

p14. Find the area of the cyclic quadrilateral with side lengths given by the solutions to

$$x^4 - 10x^3 + 34x^2 - 45x + 19 = 0.$$

p15. Suppose that we know $u_{0,m} = m^2 + m$ and $u_{1,m} = m^2 + 3m$ for all integers m, and that

$$u_{n-1,m} + u_{n+1,m} = u_{n,m-1} + u_{n,m+1}$$

Find $u_{30,-5}$.

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