Caltech Harvey Mudd Math Competition from Fall 2015

www.artofproblemsolving.com/community/c3015725

by parmenides51

–	Team Round
1	3 players take turns drawing lines that connect vertices of a regular n-gon. No player may draw a line that intersects another line at a point other than a vertex of the $n-$gon. The last player able to draw a line wins. For how many n in the range $4 \le n \le 100$ does the first player have a winning strategy?
2	You have 4 game pieces, and you play a game against an intelligent opponent who has 6. The rules go as follows: you distribute your pieces among two points a and b, and your opponent simultaneously does as well (so neither player sees what the other is doing). You win the round if you have more pieces than them on either a or$b$, and you lose the round if you only draw or have fewer pieces on both. You play the optimal strategy, assuming your opponent will play with the strategy that beats your strategy most frequently. What proportion of the time will you win?
3	A trio of lousy salespeople charge increasing prices on tomatoes as you buy more. The first charges you x_1^1 dollars for the $x_1 th$ tomato you buy from him, the second charges x_2^2 dollars for the $x_2 th$ tomato, and the third charges x_3^3 dollars for the $x_3 th$ tomato. If you want to buy 100 tomatoes for as cheap as possible, how many should you buy from the first salesperson?
4	Let $P(x) = x^{16} - x^{15} + ... - x + 1$, and let p be a prime such that $p-1$ is divisible by 34 ($p = 103$ is an example). How many integers a between 1 and $p-1$ inclusive satisfy the property that $P(a)$ is divisible by p?
5	Felix is playing a card-flipping game. n face-down cards are randomly colored, each with equal probability of being black or red. Felix starts at the 1st card. When Felix is at the kth card, he guesses its color and then flips it over. For $k < n$, if he guesses correctly, he moves onto the $(k+1)$-th card. If he guesses incorrectly, he gains k penalty points, the cards are replaced with newly randomized face-down cards, and he moves back to card 1 to continue guessing. If Felix guesses the nth card correctly, the game ends. What is the expected number of penalty points Felix earns by the end of the game?
6	The icosahedron is a convex, regular polyhedron consisting of 20 equilateral triangle for faces. A particular icosahedron given to you has labels on each of its vertices, edges, and faces. Each minute, you uniformly at random pick one of the labels on the icosahedron. If the label is on a vertex, you remove it. If the label is on an edge, you delete the label on the edge along with any labels still on the vertices of that edge. If the label is on a face, you delete the label on the face along with any labels on the edges and vertices which make up that face. What is the expected

number of minutes that pass before you have removed all labels from the icosahedron?

7 Let I be the incenter and let Γ be the incircle of $\triangle ABC$, and let $P = \Gamma \cap BC$. Let Q denote the intersection of Γ and the line passing through P parallel to AI. Let ℓ be the tangent line to Γ at Q and let $\ell \cap AB = S$, $\ell \cap AC = R$. If $AB = 7$, $BC = 6$, $AC = 5$, what is RS?

8 Let $f(n) = \sum_{d=1}^{n} \left\lfloor \frac{n}{d} \right\rfloor$ and $g(n) = f(n) - f(n-1)$. For how many n from 1 to 100 inclusive is $g(n)$ even?

9 Let T be a 2015×2015 array containing the integers $1, 2, 3, ..., 2015^2$ satisfying the property that $T_{i,a} > T_{i,b}$ for all $a > b$ and $T_{c,j} > T_{d,j}$ for all $c > d$ where $1 \leq a, b, c, d \leq 2015$ and $T_{i,j}$ represents the entry in the i-th row and j-th column of T. How many possible values are there for the entry at $T_{5,5}$?

10 Let P be the parabola in the plane determined by the equation $y = x^2$. Suppose a circle C in the plane intersects P at four distinct points. If three of these points are $(-28, 784)$,$(-2, 4)$, and $(13, 169)$, find the sum of the distances from the focus of P to all four of the intersection points

– Tiebreaker Round

1 Call a positive integer x *n-cube-invariant* if the last n digits of x are equal to the last n digits of x^3. For example, 1 is n-cube invariant for any integer n. How many 2015-cube-invariant numbers x are there such that $x < 10^{2015}$?

2 Let $a_1 = 1$, $a_2 = 1$, and for $n \geq 2$, let

$$a_{n+1} = \frac{1}{n} a_n + a_{n-1}.$$

What is a_{12}?

3 Defi ne an n-digit pair cycle to be a number with $n^2 + 1$ digits between 1 and n with every possible pair of consecutive digits. For instance, 11221 is a 2-digit pair cycle since it contains the consecutive digits 11, 12, 22, and 21. How many 3-digit pair cycles exist?

4 The following number is the product of the divisors of n.

$$46,656,000,000$$

What is n?

– Individual Round

Individual p1. The following number is the product of the divisors of n.

$$2^6 3^3$$

What is n?

p2. Let a right triangle have the sides $AB = \sqrt{3}$, $BC = \sqrt{2}$, and $CA = 1$. Let D be a point such that $AD = BD = 1$. Let E be the point on line BD that is equidistant from D and A. Find the angle $\angle AEB$.

p3. There are twelve indistinguishable blackboards that are distributed to eight different schools. There must be at least one board for each school. How many ways are there of distributing the boards?

p4. A Nishop is a chess piece that moves like a knight on its first turn, like a bishop on its second turn, and in general like a knight on odd-numbered turns and like a bishop on even-numbered turns. A Nishop starts in the bottom-left square of a 3×3-chessboard. How many ways can it travel to touch each square of the chessboard exactly once?

p5. Let a Fibonacci Spiral be a spiral constructed by the addition of quarter-circles of radius n, where each n is a term of the Fibonacci series:

$$1, 1, 2, 3, 5, 8, ...$$

(Each term in this series is the sum of the two terms that precede it.) What is the arclength of the maximum Fibonacci spiral that can be enclosed in a rectangle of area 714, whose side lengths are terms in the Fibonacci series?

p6. Suppose that $a_1 = 1$ and

$$a_{n+1} = a_n - \frac{2}{n+2} + \frac{4}{n+1} - \frac{2}{n}$$

What is a_{15}?

p7. Consider 5 points in the plane, no three of which are collinear. Let n be the number of circles that can be drawn through at least three of the points. What are the possible values of n?

p8. Find the number of positive integers n satisfying $\lfloor n/2014 \rfloor = \lfloor n/2016 \rfloor$.

p9. Let f be a function taking real numbers to real numbers such that for all reals $x \neq 0, 1$, we have

$$f(x) + f\left(\frac{1}{1-x}\right) = (2x-1)^2 + f\left(1 - \frac{1}{x}\right)$$

Compute $f(3)$.

p10. Alice and Bob split 5 beans into piles. They take turns removing a positive number of beans from a pile of their choice. The player to take the last bean loses. Alice plays first. How many ways are there to split the piles such that Alice has a winning strategy?

p11. Triangle ABC is an equilateral triangle of side length 1. Let point M be the midpoint of side AC. Another equilateral triangle DEF, also of side length 1, is drawn such that the circumcenter of DEF is M, point D rests on side AB. The length of AD is of the form $\frac{a+\sqrt{b}}{c}$, where b is square free. What is $a+b+c$?

p12. Consider the function $f(x) = \max\{-11x-37, x-1, 9x+3\}$ defined for all real x. Let $p(x)$ be a quadratic polynomial tangent to the graph of f at three distinct points with x values t_1, t_2 and t_3 Compute the maximum value of $t_1+t_2+t_3$ over all possible p.

p13. Circle J_1 of radius 77 is centered at point X and circle J_2 of radius 39 is centered at point Y. Point A lies on $J1$ and on line XY , such that A and Y are on opposite sides of X. Ω is the unique circle simultaneously tangent to the tangent segments from point A to J_2 and internally tangent to J_1. If $XY = 157$, what is the radius of Ω ?

p14. Find the smallest positive integer n so that for any integers $a_1, a_2, ..., a_{527}$,the number

$$\left(\prod_{j=1}^{527} a_j\right) \cdot \left(\prod_{j=1}^{527} a_j^n\right)$$

is divisible by 527.

p15. A circle Ω of unit radius is inscribed in the quadrilateral $ABCD$. Let circle ω_A be the unique circle of radius r_A externally tangent to Ω, and also tangent to segments AB and DA. Similarly define circles ω_B, ω_C, and ω_D and radii r_B, r_C, and r_D. Compute the smallest positive real λ so that $r_C < \lambda$ over all such configurations with $r_A > r_B > r_C > r_D$.

PS. You had better use hide for answers. Collected here (`https://artofproblemsolving.com/community/c5h2760506p24143309`).