**Utah Mathematical Olympiad 2022**

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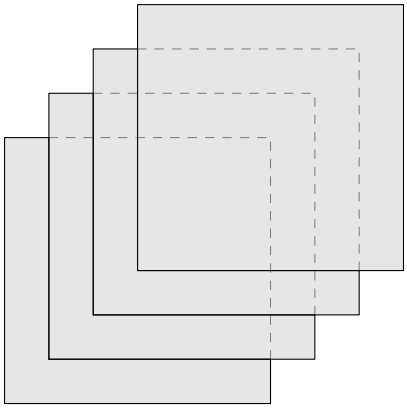
by kred9

**1** Let $n \geq 2$ be an integer. Thibaud the Tiger lays $n$ $2 \times 2$ overlapping squares out on a table, such that the centers of the squares are equally spaced along the line $y = x$ from $(0, 0)$ to $(1, 1)$ (including the two endpoints). For example, for $n = 4$ the resulting figure is shown below, and it covers a total area of $\frac{23}{3}$.



Find, with proof, the minimum $n$ such that the figure covers an area of at least $\sqrt{63}$.

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**2** Let $x$ and $y$ be relatively prime integers. Show that $x^2 + xy + y^2$ and $x^2 + 3xy + y^2$ are relatively prime.

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**3** Find all sequences $a_1, a_2, a_3, \ldots$ of real numbers such that for all positive integers $m, n \geq 1$, we have

$$a_{m+n} = a_m + a_n - mn \text{ and}$$
$$a_{mn} = m^2 a_n + n^2 a_m + 2a_m a_n.$$

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**4** Alpha and Beta are playing a game on a $10 \times 100$ grid of squares. At each turn, they can fold the grid along any of the interior horizontal or vertical gridlines, which creates a smaller (folded) grid of squares (on the first move, they can choose one of $9$ horizontal or $99$ vertical gridlines). The

 

person who makes the last fold wins. If both players play optimally and Alpha starts, determine with proof who wins.

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**5** $2022$ lily pads are arranged in a circle. Each lily pad starts with height $1$. A frog starts on one of the lily pads, and jumps around clockwise as follows: if the frog is on a lily bad of height $k$, the lily pad grows by $1$ (becoming $k+1$), and then the frog jumps $k$ lily pads clockwise (i.e. jumping over $(k-1)$). The frog continues doing this as long as it pleases.

After $n$ jumps, let $D(n)$ be the difference between the tallest lily pad and the shortest lily pad. Find, with proof, the maximum possible value of $D(n)$, or prove that $D(n)$ is unbounded.

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**6** An $m \times n$ grid of squares (with $m$ rows and $n$ columns) has some of its squares colored blue. The grid is called *fish-friendly* if a fish can swim from the left edge of the grid to the right edge of the grid only moving through blue squares. In other words, there is a sequence of blue squares, each horizontally or vertically adjacent to the previous square, starting in the first column and ending in the last column.

Prove that the number of fish-friendly $42 \times 49$ grids is at least $2^{2022}$.

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