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Problem 8.1 Let $P=(x^4-40x^2+144)(x^3-16x)$. $a)$ Factor P as a product of irreducible polynomials. $b)$ We write down the values of $P(10)$ and $P(91)$. What is the greatest common divisor of the two numbers?

Problem 8.2 Let $\triangle ABC$ have $AB=1$ cm, $BC=2$ cm and $AC=\sqrt{3}$ cm. Points D, E and F lie on segments AB, AC and BC respectively are such that $AE=BD$ and $BF=AD$. The angle bisector of $\angle BAC$ intersects the circumcircle of $\triangle ADE$ for the second time at M and the angle bisector of $\angle ABC$ intersects the circumcircle of $\triangle BDF$ at N. Determine the length of MN.

Problem 8.3 Given the inequalities: $a)$ $\left(\frac{2a}{b+c}\right)^2+\left(\frac{2b}{a+c}\right)^2+\left(\frac{2c}{a+b}\right)^2\geq\frac{a}{c}+\frac{b}{a}+\frac{c}{b}$ $b)$ $\left(\frac{a+b}{c}\right)^2+\left(\frac{b+c}{a}\right)^2+\left(\frac{c+a}{b}\right)^2\geq\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+9$
For each of them either prove that it holds for all positive real numbers a, b, c or present a counterexample (a,b,c) which doesn't satisfy the inequality.

Problem 8.4 Let $p=(a_1,a_2,\ldots,a_{12})$ be a permutation of $1,2,\ldots,12$.
We will denote

$$S_p=|a_1-a_2|+|a_2-a_3|+\ldots+|a_{11}-a_{12}|$$

We'll call p *optimistic* if $a_i>\min(a_{i-1},a_{i+1})\ \forall i=2,\ldots,11$. $a)$ What is the maximum possible value of S_p. How many permutations p achieve this maximum?
$b)$ What is the number of *optimistic* permtations p? $c)$ What is the maximum possible value of S_p for an *optimistic* p? How many *optimistic* permutations p achieve this maximum?

Problem 9.1 Let $f(x)$ be a quadratic function with integer coefficients. If we know that $f(0)$, $f(3)$ and $f(4)$ are all different and elements of the set $\{2,20,202,2022\}$, determine all possible values of $f(1)$.

Problem 9.2 Let $\triangle ABC$ have median CM ($M\in AB$) and circumcenter O. The circumcircle of $\triangle AMO$ bisects CM. Determine the least possible perimeter of $\triangle ABC$ if it has integer side lengths.

Problem 9.3 Find all primes p, such that there exist positive integers x, y which satisfy

$$\begin{cases} p+49=2x^2\\ p^2+49=2y^2\end{cases}$$

Problem 9.4 14 students attend the IMO training camp. Every student has at least k favourite numbers. The organisers want to give each student a shirt with one of the student's favourite numbers on the back. Determine the least k, such that this is always possible if: $a)$ The students can be arranged in a circle such that every two students sitting next to one another have different numbers. $b)$ 7 of the students are boys, the rest are girls, and there isn't a boy and a girl with the same number.

Problem 10.1 If $x, y, z \in \mathbb{R}$ are solutions to the system of equations

$$\begin{cases} x-y+z-1=0 \\ xy+2z^2-6z+1=0 \end{cases}$$

what is the greatest value of $(x-1)^2+(y+1)^2$?

Problem 10.2 Let $\triangle ABC$ have incenter I. The line CI intersects the circumcircle of $\triangle ABC$ for the second time at L, and $CI = 2IL$. Points M and N lie on the segment AB, such that $\angle AIM = \angle BIN = 90^\circ$. Prove that $AB = 2MN$.

Problem 10.3 A permutation σ of the numbers $1, 2, \dots, 10$ is called *bad* if there exist integers i, j, k which satisfy

$$1 \le i < j < k \le 10 \quad \text{and} \quad \sigma(j) < \sigma(k) < \sigma(i)$$

and *good* otherwise. Find the number of *good* permutations.

Problem 10.4 Find the smallest odd prime p, such that there exist coprime positive integers k and ℓ which satisfy

$$4k-3\ell=12 \quad \text{and} \quad \ell^2+\ell k+k^2 \equiv 3 \pmod{p}$$

Problem 11.1 Solve the equation

$$(x+1)\log_3^2 x + 4x\log_3 x - 16 = 0$$

Problem 11.2 A circle through the vertices A and B of $\triangle ABC$ intersects segments AC and BC at points P and Q respectively. If $AQ = AC$, $\angle BAQ = \angle CBP$ and $AB = (\sqrt{3}+1)PQ$, find the measures of the angles of $\triangle ABC$.

Problem 11.3 In every cell of a table with n rows and m columns is written one of the letters a, b, c. Every two rows of the table have the same letter in at most $k \ge 0$ positions and every two columns coincide at most k positions. Find m, n, k if

$$\frac{2mn+6k}{3(m+n)} \ge k+1$$

Problem 11.4 Let $n \geq 2$ be a positive integer. The set M consists of $2n^2 - 3n + 2$ positive rational numbers. Prove that there exists a subset A of M with n elements with the following property: $\forall\, 2 \leq k \leq n$ the sum of any k (not necessarily distinct) numbers from A is not in A.

Problem 12.1 $ABCD$ is circumscribed in a circle k, such that $[ACB] = s$, $[ACD] = t$, $s < t$. Determine the smallest value of $\frac{4s^2+t^2}{5st}$ and when this minimum is achieved.

Problem 12.2 Let $ABCDV$ be a regular quadrangular pyramid with V as the apex. The plane λ intersects the VA, VB, VC and VD at M, N, P, Q respectively. Find $VQ : QD$, if $VM : MA = 2 : 1$, $VN : NB = 1 : 1$ and $VP : PC = 1 : 2$.

Problem 12.3 Let $P, Q \in \mathbb{R}[x]$, such that Q is a 2021-degree polynomial and let $a_1, a_2, \ldots, a_{2022}, b_1, b_2, \ldots, b_{2022}$ be real numbers such that $a_1 a_2 \ldots a_{2022} \neq 0$. If for all real x

$$P(a_1 Q(x) + b_1) + \ldots + P(a_{2021} Q(x) + b_{2021}) = P(a_{2022} Q(x) + b_{2022})$$

prove that $P(x)$ has a real root.

Problem 12.4 Let m and n be positive integers and p be a prime number. Find the greatest positive integer s (as a function of m, n and p) such that from a random set of mnp positive integers we can choose snp numbers, such that they can be partitioned into s sets of np numbers, such that the sum of the numbers in every group gives the same remainder when divided by p.
