Vietnam National Olympiad 2016

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– Day 1

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1 Solve the system of equations $\begin{cases} 6x-y+z^2=3 \\ x^2-y^2-2z=-1 \qquad (x,y,z\in\mathbb{R}.) \\ 6x^2-3y^2-y-2z^2=0 \end{cases}$.

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2 a) Let (a_n) be the sequence defined by $a_n=\ln(2n^2+1)-\ln(n^2+n+1)\ \forall n\geq 1$. Prove that the set $\{n\in\mathbb{N}|\ \{a_n\}<\dfrac{1}{2}\}$ is a finite set;
b) Let (b_n) be the sequence defined by $a_n=\ln(2n^2+1)+\ln(n^2+n+1)\ \forall n\geq 1$. Prove that the set $\{n\in\mathbb{N}|\ \{b_n\}<\dfrac{1}{2016}\}$ is an infinite set.

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3 Let ABC be an acute triange with B,C fixed. Let D be the midpoint of BC and E,F be the foot of the perpendiculars from D to AB,AC, respectively.
a) Let O be the circumcenter of triangle ABC and $M=EF\cap AO, N=EF\cap BC$. Prove that the circumcircle of triangle AMN passes through a fixed point;
b) Assume that tangents of the circumcircle of triangle AEF at E,F are intersecting at T. Prove that T is on a fixed line.

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4 Let m and n be positive integers. A people planted two kind of different trees on a plot tabular grid size $m\times n$ (each square plant one tree.) A plant called *inpressive* if two conditions following conditions are met simultaneously:
i) The number of trees in each of kind is equal;
ii) In each row the number of tree of each kind is diffrenent not less than a half of number of cells on that row and In each colum the number of tree of each kind is diffrenent not less than a half of number of cells on that colum.

a) Find an inpressive plant when $m=n=2016$;
b) Prove that if there at least a inpressive plant then $4|m$ and $4|n$.

– Day 2

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1 Find all $a\in\mathbb{R}$ such that there is function $f:\mathbb{R}\to\mathbb{R}$
i) $f(1)=2016$
ii) $f(x+y+f(y))=f(x)+ay \quad \forall x,y\in\mathbb{R}$

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2 Given a triangle ABC inscribed by circumcircle (O). The angles at B, C are acute angle. Let M on the arc BC that doesn't contain A such that AM is not perpendicular to BC. AM meets the perpendicular bisector of BC at T. The circumcircle (AOT) meets (O) at N ($N \neq A$).

a) Prove that $\angle BAM = \angle CAN$.

b) Let I be the incenter and G be the foor of the angle bisector of $\angle BAC$. AI, MI, NI intersect (O) at D, E, F respectively. Let $P = DF \cap AM, Q = DE \cap AN$. The circle passes through P and touches AD at I meets DF at H ($H \neq D$).The circle passes through Q and touches AD at I meets DE at K ($K \neq D$). Prove that the circumcircle (GHK) touches BC.

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3 a) Prove that if n is an odd perfect number then n has the following form

$$n = p^s m^2$$

where p is prime has form $4k + 1$, s is positive integers has form $4h + 1$, and $m \in \mathbb{Z}^+$, m is not divisible by p.

b) Find all $n \in \mathbb{Z}^+$, $n > 1$ such that $n - 1$ and $\frac{n(n+1)}{2}$ is perfect number

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