District Olympiad 2022

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–	Grade 9
P1	Let $f:\mathbb{N}^*\to\mathbb{N}^*$ be a function such that $\frac{x^3+3x^2f(y)}{x+f(y)}+\frac{y^3+3y^2f(x)}{y+f(x)}=\frac{(x+y)^3}{f(x+y)},~(\forall)x,y\in\mathbb{N}^*.$ $a)$ Prove that $f(1)=1.$ $b)$ Find function $f.$
P2	$a)$ Prove that $2x^3-3x^2+1\geq 0,~(\forall)x\geq 0.$ $b)$ Let $x,y,z\geq 0$ such that $\frac{2}{1+x^3}+\frac{2}{1+y^3}+\frac{2}{1+z^3}=3.$ Prove that $\frac{1-x}{1-x+x^2}+\frac{1-y}{1-y+y^2}+\frac{1-z}{1-z+z^2}\geq 0.$
P3	$a)$ Solve over the positive integers $3^x=x+2.$ $b)$ Find pairs $(x,y)\in\mathbb{N}\times\mathbb{N}$ such that $(x+3^y)$ and $(y+3^x)$ are consecutive.
P4	We call a set of 6 points in the plane *splittable* if we if can denote its elements by A,B,C,D,E and F in such a way that $\triangle ABC$ and $\triangle DEF$ have the same centroid. -Construct a splittable set. -Show that any set of 7 points has a subset of 6 points which is *not* splittable.
–	Grade 10
P1	Determine all $x\in (0,3/4)$ which satisfy $$\log_x(1-x)+\log_2\frac{1-x}{x}=\frac{1}{(\log_2x)^2}.$$
P2	Let z_1,z_2 and z_3 be complex numbers of modulus 1, such that $\vert z_i-z_j\vert\geq\sqrt{2}$ for all $i\neq j\in\{1,2,3\}$. Prove that $$\vert z_1+z_2\vert+\vert z_2+z_3\vert+\vert z_3+z_2\vert\leq 3.$$ *Mathematical Gazette*
P3	A positive integer $n\geq 4$ is called *interesting* if there exists a complex number z such that $\vert z\vert=1$ and $$1+z+z^2+z^{n-1}+z^n=0.$$ Find how many interesting numbers are smaller than $2022.$

P4 Find all functions $f:\mathbb{R}\to\mathbb{R}$ which satisfy the following relationship for all $x,y\in\mathbb{R}:$

$$f(f(y-x)-xf(y))+f(x)=y\cdot(1-f(x)).$$

– Grade 11

P1 Let $f,g:\mathbb{R}\to\mathbb{R}$ be functions which satisfy

$$\inf_{x>a} f(x)=g(a) \text{ and } \sup_{x<a} g(x)=f(a),$$

for all $a\in\mathbb{R}$. Given that f has Darboux's Property (intermediate value property), show that functions f and g are continuous and equal to each other.

Mathematical Gazette

P2 Let $A,B\in\mathcal{M}_3(\mathbb{R})$ de matrices such that $A^2+B^2=O_3$. Prove that $\det(aA+bB)=0$ for any real numbers a and b.

P3 Let $(x_n)_{n\geq 1}$ be the sequence defined recursively as such:

$$x_1=1,\ x_{n+1}=\frac{x_1}{n+1}+\frac{x_2}{n+2}+\cdots+\frac{x_n}{2n}\ \forall n\geq 1.$$

Consider the sequence $(y_n)_{n\geq 1}$ such that $y_n=(x_1^2+x_2^2+\cdots x_n^2)/n$ for all $n\geq 1$. Prove that

-$x_{n+1}^2<y_n/2$ and $y_{n+1}<(2n+1)/(2n+2)\cdot y_n$ for all $n\geq 1$;
-$\lim_{n\to\infty} x_n=0$.

P4 Let $A\in\mathcal{M}_n(\mathbb{C})$ where $n\geq 2$. Prove that if $m=|\{\text{rank}(A^k)-\text{rank}(A^{k+1})\text{"}\ :\ k\in\mathbb{N}^*\}|$ then $n+1\geq m(m+1)/2$.

– Grade 12

P1 Let e be the identity of monoid $(M,\cdot)$ and $a\in M$ an invertible element. Prove that

-The set $M_a:=\{x\in M: ax^2a=e\}$ is nonempty;
-If $b\in M_a$ is invertible, then $b^{-1}\in M_a$ if and only if $a^4=e$;
-If $(M_a,\cdot)$ is a monoid, then $x^2=e$ for all $x\in M_a$.

Mathematical Gazette

P2 Let $(G,\cdot)$ be a group and $H\neq G$ be a subgroup so that $x^2=y^2$ for all $x,y\in G\setminus H$. Show that $(H,\cdot)$ is an Abelian group.

P3 Find all values of $n \in \mathbb{N}^*$ for which

$$I_n := \int_0^{\pi} \cos(x) \cdot \cos(2x) \cdot \ldots \cdot \cos(nx)\, dx = 0.$$

P4 Let $I \subseteq \mathbb{R}$ be an open interval and $f : I \to \mathbb{R}$ a strictly monotonous function. Prove that for all $c \in I$ there exist $a, b \in I$ such that $c \in (a, b)$ and

$$\int_a^b f(x)\, dx = f(c) \cdot (b - a).$$
