Finals 2022

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–	Day 1
1	Let ABC be an acute triangle with $AB < AC$. The angle bisector of BAC intersects the side BC and the circumcircle of ABC at D and $M \neq A$, respectively. Points X and Y are chosen so that $MX \perp AB$, $BX \perp MB$, $MY \perp AC$, and $CY \perp MC$. Prove that the points X, D, Y are collinear.
2	Let $m, n \geq 2$ be given integers. Prove that there exist positive integers $a_1 < a_2 < \ldots < a_m$ so that for any $1 \leq i < j \leq m$ the number $\frac{a_j}{a_j - a_i}$ is an integer divisible by n.
3	One has marked n points on a circle and has drawn a certain number of chords whose endpoints are the marked points. It turned out that the following property is satisfied: whenever any 2021 drawn chords are removed one can join any two marked points by a broken line composed of some of the remaining drawn chords. Prove that one can remove some of the drawn chords so that at most $2022n$ chords remain and the property described above is preserved.
–	Day 2
4	Find all triples (a, b, c) of real numbers satisfying the system $\begin{cases} a^3 + b^2c = ac \\ b^3 + c^2a = ba \\ c^3 + a^2b = cb \end{cases}$
5	Let ABC be a triangle satisfying $AB < AC$. Let M be the midpoint of BC. A point P lies on the segment AB with $AP > PB$. A point Q lies on the segment AC with $\angle MPA = \angle AQM$. The perpendicular bisectors of BC and PQ intersect at S. Prove that $\angle BAC + \angle QSP = \angle QMP$.
6	A prime number p and a positive integer n are given. Prove that one can colour every one of the numbers $1, 2, \ldots, p-1$ using one of the $2n$ colours so that for any $i = 2, 3, \ldots, n$ the sum of any i numbers of the same colour is not divisible by p.