Moldova Team Selection Test 2022

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– Day 1

1 Show that for every integer $n \geq 2$ the number

$$a = n^{5n-1} + n^{5n-2} + n^{5n-3} + n + 1$$

is a composite number.

2 Real numbers a, b, c, d satisfy

$$a^2 + b^2 + c^2 + d^2 = 4.$$

Find the greatest possible value of

$$E(a,b,c,d) = a^4 + b^4 + c^4 + d^4 + 4(a+b+c+d)^2.$$

3 Let n be a positive integer. On a board there are written all integers from 1 to n. Alina does n moves consecutively: for every integer m $(1 \leq m \leq n)$ the move m consists in changing the sign of every number divisible by m. At the end Alina sums the numbers. Find this sum.

4 In the acute triangle ABC the point M is on the side BC. The inscribed circle of triangle ABM touches the sides BM, MA and AB in points D, E and F, and the inscribed circle of triangle ACM touches the sides CM, MA and AC in points X, Y and Z. The lines FD and ZX intersect in point H. Prove that lines AH, XY and DE are concurrent.

– Day 2

5 The function $f : \mathbb{N} \to \mathbb{N}$ verifies: $1) f(n+2) - 2022 \cdot f(n+1) + 2021 \cdot f(n) = 0, \forall n \in \mathbb{N}$; $2) f(20^{22}) = f(22^{20})$; $3) f(2021) = 2022$.
Find all possible values of $f(2022)$.

6 Let A be a point outside of the circle Ω. Tangents from A touch Ω in points B and C. Point C, collinear with A and P, is between A and P, such that the circumcircle of triangle ABP intersects Ω again in point E. Point Q is on the segment BP, such that $\angle PEQ = 2 \cdot \angle APB$. Prove that the lines BP and CQ are perpendicular.

7 Let $f : \mathbb{N} \to \mathbb{N}$, $f(n) = n^2 - 69n + 2250$ be a function. Find the prime number p, for which the sum of the digits of the number $f(p^2 + 32)$ is as small as possible.

8 a) Let n $(n \geq 2)$ be an integer. On a line there are n distinct (pairwise distinct) sets of points, such that for every integer k $(1 \leq k \leq n)$ the union of every k sets contains exactly $k+1$ points. Show that there is always a point that belongs to every set.
b) Is the same conclusion true if there is an infinity of distinct sets of points such that for every positive integer k the union of every k sets contains exactly $k+1$ points?

– Day 3

9 Let n be a positive integer. A grid of dimensions $n \times n$ is divided in n^2 1×1 squares. Every segment of length 1 (side of a square) from this grid is coloured in blue or red. The number of red segments is not greater than n^2. Find all positive integers n, for which the grid always will cointain at least one 1×1 square which has at least three blue sides.

10 Let $P(X)$ be a polynomial with positive coefficients. Show that for every integer $n \geq 2$ and every n positive numbers $x_1, x_2, ..., x_n$ the following inequality is true:

$$P\left(\frac{x_1}{x_2}\right)^2 + P\left(\frac{x_2}{x_3}\right)^2 + ... + P\left(\frac{x_n}{x_1}\right)^2 \geq n \cdot P(1)^2.$$

When does the equality take place?

11 Let Ω be the circumcircle of triangle ABC such that the tangents to Ω in points B and C intersect in P. The squares ABB_1B_2 and ACC_1C_2 are constructed on the sides AB and AC in the exterior of triangle ABC, such that the lines B_1B_2 and C_1C_2 intersect in point Q. Prove that P, A, and Q are collinear.

12 Let $(x_n)_{n\geq 1}$ be a sequence that verifies:

$$x_1 = 1, \quad x_2 = 7, \quad x_{n+1} = x_n + 3x_{n-1}, \forall n \geq 2.$$

Prove that for every prime number p the number $x_p - 1$ is divisible by $3p$.