Iran Team Selection Test 2022

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by TheBarioBario

1 Morteza Has 100 sets. at each step Mahdi can choose two distinct sets of them and Morteza tells him the intersection and union of those two sets. Find the least steps that Mahdi can find all of the sets.

Proposed by Morteza Saghafian

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2 For a positive integer n, let $\tau(n)$ and $\sigma(n)$ be the number of positive divisors of n and the sum of positive divisors of n, respectively. let a and b be positive integers such that $\sigma(a^n)$ divides $\sigma(b^n)$ for all $n \in \mathbb{N}$. Prove that each prime factor of $\tau(a)$ divides $\tau(b)$.

Proposed by MohammadAmin Sharifi

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3 Incircle ω of triangle ABC is tangent to sides CB and CA at D and E, respectively. Point X is the reflection of D with respect to B. Suppose that the line DE is tangent to the A-excircle at Z. Let the circumcircle of triangle XZE intersect ω for the second time at K. Prove that the intersection of BK and AZ lies on ω.

Proposed by Mahdi Etesamifard and Alireza Dadgarnia

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4 Cyclic quadrilateral $ABCD$ with circumcenter O is given. Point P is the intersection of diagonals AC and BD. Let M and N be the midpoint of the sides AD and BC, respectively. Suppose that ω_1, ω_2 and ω_3 be the circumcircle of triangles ADP, BCP and OMN, respectively. The intersection point of ω_1 and ω_3, which is not on the arc APD of ω_1, is E and the intersection point of ω_2 and ω_3, which is not on the arc BPC of ω_2, is F. Prove that $OF = OE$.

Proposed by Seyed Amirparsa Hosseini Nayeri

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5 Find all $C \in \mathbb{R}$ such that every sequence of integers $\{a_n\}_{n=1}^{\infty}$ which is bounded from below and for all $n \geq 2$ satisfy

$$0 \leq a_{n-1} + Ca_n + a_{n+1} < 1$$

is periodic.

Proposed by Navid Safaei

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6 Let m, n and $a_1, a_2, \ldots, a_m$ be arbitrary positive integers. Ali and Mohammad Play the following game. At each step, Ali chooses $b_1, b_2, \ldots, b_m \in \mathbb{N}$ and then Mohammad chosses a positive integers s and obtains a new sequence $\{c_i = a_i + b_{i+s}\}_{i=1}^{m}$, where

$$b_{m+1} = b_1,\ b_{m+2} = b_2, \ldots,\ b_{m+s} = b_s$$

The goal of Ali is to make all the numbers divisible by n in a finite number of steps. FInd all positive integers m and n such that Ali has a winning strategy, no matter how the initial values $a_1, a_2, \dots, a_m$ are.
after we create the c_i s, this sequence becomes the sequence that we continue playing on, as in it is our 'new' a_i

Proposed by Shayan Gholami

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7 Suppose that n is a positive integer number. Consider a regular polygon with $2n$ sides such that one of its largest diagonals is parallel to the x-axis. Find the smallest integer d such that there is a polynomial P of degree d whose graph intersects all sides of the polygon on points other than vertices.

Proposed by Mohammad Ahmadi

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8 In triangle ABC, with $AB < AC$, I is the incenter, E is the intersection of A-excircle and BC. Point F lies on the external angle bisector of BAC such that E and F lieas on the same side of the line AI and $\angle AIF = \angle AEB$. Point Q lies on BC such that $\angle AIQ = 90$. Circle ω_b is tangent to FQ and AB at B, circle ω_c is tangent to FQ and AC at C and both circles pass through the inside of triangle ABC. if M is the Midpoint od the arc BC, which does not contain A, prove that M lies on the radical axis of ω_b and ω_c.

Proposed by Amirmahdi Mohseni

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9 consider $n \geq 6$ points $x_1, x_2, \dots, x_n$ on the plane such that no three of them are colinear. We call graph with vertices $x_1, x_2, \dots, x_n$ a "road network" if it is connected, each edge is a line segment, and no two edges intersect each other at points other than the vertices. Prove that there are three road networks G_1, G_2, G_3 such that G_i and G_j don't have a common edge for $1 \leq i, j \leq 3$.

Proposed by Morteza Saghafian

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10 We call an infinite set $S \subseteq \mathbb{N}$ good if for all parwise different integers $a, b, c \in S$, all positive divisors of $\frac{a^c - b^c}{a - b}$ are in S. for all positive integers $n > 1$, prove that there exists a good set S such that $n \notin S$.

Proposed by Seyed Reza Hosseini Dolatabadi

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11 Consider a table with n rows and $2n$ columns. we put some blocks in some of the cells. After putting blocks in the table we put a robot on a cell and it starts moving in one of the directions right, left, down or up. It can change the direction only when it reaches a block or border. Find the smallest number m such that we can put m blocks on the table and choose a starting point for the robot so it can visit all of the unblocked cells. (the robot can't enter the blocked cells.)

Proposed by Seyed Mohammad Seyedjavadi and Alireza Tavakoli

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12 suppose that A is the set of all Closed intervals $[a,b] \subset \mathbb{R}$. Find all functions $f : \mathbb{R} \rightarrow A$ such that • $x \in f(y) \Leftrightarrow y \in f(x)$ • $|x-y| > 2 \Leftrightarrow f(x) \cap f(y) = \varnothing$ • For all real numbers $0 \leq r \leq 1$, $f(r) = [r^2 - 1, r^2 + 1]$

Proposed by Matin Yousefi