Serbia National Math Olympiad 2022

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–	Day1
P1	Let k be incircle of acute triangle ABC, $AC \neq BC$, and l be excircle that touches AB. Line p through the C is orthogonal to AB, $p \cap k = \{X, Y\}$, $p \cap l = \{Z, T\}$ and the point arrangement is $X - Y - Z - T$. Circle m through X and Z intersects AB at D and E. Prove that points D, Y, E, T are concyclic.
P2	Let a, b and c be positive real numbers and $a^3 + b^3 + c^3 = 3$. Prove $$\frac{1}{3-2a} + \frac{1}{3-2b} + \frac{1}{3-2c} \geq 3$$
P3	The table of dimensions $n \times n$, $n \in \mathbb{N}$, is filled with numbers from 1 to n^2, but the difference any two numbers on adjacent fields is at most n, and that for every $k = 1, 2, \ldots, n^2$ set of fields whose numbers are $1, 2, \ldots, k$ is connected, as well as the set of fields whose numbers are $k, k+1, \ldots, n^2$. Neighboring fields are fields with a common side, while a set of fields is considered connected if from each field to every other field of that set can be reached going only to the neighboring fields within that set. We call a pair of adjacent numbers, ie. numbers on adjacent fields, good, if their absolute difference is exactly n (one number can be found in several good pairs). Prove that the table has at least $2(n-1)$ good pairs.
–	Day2
P4	Let $f(n)$ be number of numbers $x \in \{1, 2, \cdots, n\}$, $n \in \mathbb{N}$, such that $gcd(x, n)$ is either 1 or prime. Prove $$\sum_{d\mid n} f(d) + \varphi(n) \geq 2n$$ For which n does equality hold?
P5	On the board are written n natural numbers, $n \in \mathbb{N}$. In one move it is possible to choose two equal written numbers and increase one by 1 and decrease the other by 1. Prove that in this the game cannot be played more than $\frac{n^3}{6}$ moves.

P6 Let p and q be different primes, and $\alpha \in (0,3)$ a real number. Prove that in sequence

$$[\alpha], [2\alpha], [3\alpha] \dots$$

exists number less than $2pq$, divisible by p or q.