EGMO 2022

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– Day 1

1 Let ABC be an acute-angled triangle in which $BC < AB$ and $BC < CA$. Let point P lie on segment AB and point Q lie on segment AC such that $P \neq B$, $Q \neq C$ and $BQ = BC = CP$. Let T be the circumcenter of triangle APQ, H the orthocenter of triangle ABC, and S the point of intersection of the lines BQ and CP. Prove that T, H, and S are collinear.

2 Let $\mathbb{N} = \{1, 2, 3, \dots\}$ be the set of all positive integers. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that for any positive integers a and b, the following two conditions hold:
(1) $f(ab) = f(a)f(b)$, and
(2) at least two of the numbers $f(a)$, $f(b)$, and $f(a + b)$ are equal.

3 An infinite sequence of positive integers $a_1, a_2, \dots$ is called $good$ if
(1) a_1 is a perfect square, and
(2) for any integer $n \geq 2$, a_n is the smallest positive integer such that

$$na_1 + (n-1)a_2 + \dots + 2a_{n-1} + a_n$$

is a perfect square.
Prove that for any good sequence $a_1, a_2, \dots$, there exists a positive integer k such that $a_n = a_k$ for all integers $n \geq k$.
(reposting because the other thread didn't get moved)

– Day 2

4 Given a positive integer $n \geq 2$, determine the largest positive integer N for which there exist $N + 1$ real numbers $a_0, a_1, \dots, a_N$ such that $(1)\ \ a_0 + a_1 = -\frac{1}{n}$, and $(2)\ \ (a_k + a_{k-1})(a_k + a_{k+1}) = a_{k-1} - a_{k+1}$ for $1 \leq k \leq N - 1$.

5 For all positive integers n, k, let $f(n, 2k)$ be the number of ways an $n \times 2k$ board can be fully covered by nk dominoes of size 2×1. (For example, $f(2, 2) = 2$ and $f(3, 2) = 3$.) Find all positive integers n such that for every positive integer k, the number $f(n, 2k)$ is odd.

6 Let $ABCD$ be a cyclic quadrilateral with circumcenter O. Let the internal angle bisectors at A and B meet at X, the internal angle bisectors at B and C meet at Y, the internal angle bisectors at C and D meet at Z, and the internal angle bisectors at D and A meet at W. Further, let AC and BD meet at P. Suppose that the points X, Y, Z, W, O, and P are distinct.

Prove that O, X, Y, Z, W lie on the same circle if and only if P, X, Y, Z, and W lie on the same circle.