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– Day 1

1 Let x, y, z be positive reals such that $xyz = 1$. Prove that
$$\sum_{cyc} \frac{1}{\sqrt{x+2y+6}} \le \sum_{cyc} \frac{x}{\sqrt{x^2+4\sqrt{y}+4\sqrt{z}}}.$$

2 A positive integer $n < 2017$ is given. Exactly n vertices of a regular 2017-gon are colored red, and the remaining vertices are colored blue. Prove that the number of isosceles triangles whose vertices are monochromatic does not depend on the chosen coloring (but does depend on n.)

4 Let $\triangle ABC$ be an acute triangle with altitudes AA_1, BB_1, CC_1 and orthocenter H. Let K, L be the midpoints of BC_1, CB_1. Let ℓ_A be the external angle bisector of $\angle BAC$. Let ℓ_B, ℓ_C be the lines through B, C perpendicular to ℓ_A. Let ℓ_H be the line through H parallel to ℓ_A. Prove that the centers of the circumcircles of $\triangle A_1B_1C_1, \triangle AKL$ and the rectangle formed by $\ell_A, \ell_B, \ell_C, \ell_H$ lie on the same line.

3 Does there exist an arithmetic progression with 2017 terms such that each term is not a perfect power, but the product of all 2017 terms is?

– Day 2

1 Determine all integers $n \geq 2$ having the following property: for any integers $a_1, a_2, \ldots, a_n$ whose sum is not divisible by n, there exists an index $1 \le i \le n$ such that none of the numbers
$$a_i, a_i + a_{i+1}, \ldots, a_i + a_{i+1} + \ldots + a_{i+n-1}$$
is divisible by n. Here, we let $a_i = a_{i-n}$ when $i > n$.

Proposed by Warut Suksompong, Thailand

2 For finite sets A, M such that $A \subseteq M \subset \mathbb{Z}^+$, we define
$$f_M(A) = \{x \in M \mid x \text{ is divisible by an odd number of elements of } A\}.$$
Given a positive integer k, we call M *k-colorable* if it is possible to color the subsets of M with k colors so that for any $A \subseteq M$, if $f_M(A) \neq A$ then $f_M(A)$ and A have different colors.
Determine the least positive integer k such that every finite set $M \subset \mathbb{Z}^+$ is k-colorable.

3 Let S be a finite set, and let $\mathcal{A}$ be the set of all functions from S to S. Let f be an element of $\mathcal{A}$, and let $T = f(S)$ be the image of S under f. Suppose that $f \circ g \circ f \neq g \circ f \circ g$ for every g in $\mathcal{A}$ with $g \neq f$. Show that $f(T) = T$.

– Day 3

1 A rectangle $\mathcal{R}$ with odd integer side lengths is divided into small rectangles with integer side lengths. Prove that there is at least one among the small rectangles whose distances from the four sides of $\mathcal{R}$ are either all odd or all even.

Proposed by Jeck Lim, Singapore

2 Let O be the circumcenter of an acute triangle ABC. Line OA intersects the altitudes of ABC through B and C at P and Q, respectively. The altitudes meet at H. Prove that the circumcenter of triangle PQH lies on a median of triangle ABC.

3 Find the smallest positive integer n or show no such n exists, with the following property: there are infinitely many distinct n-tuples of positive rational numbers $(a_1, a_2, \ldots, a_n)$ such that both

$$a_1 + a_2 + \cdots + a_n \quad \text{and} \quad \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}$$

are integers.

– Day 4

1 Let $a_1, a_2, \ldots a_n, k$, and M be positive integers such that

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} = k \quad \text{and} \quad a_1 a_2 \cdots a_n = M.$$

If $M > 1$, prove that the polynomial

$$P(x) = M(x+1)^k - (x+a_1)(x+a_2)\cdots(x+a_n)$$

has no positive roots.

2 Let $(x_1, x_2, \ldots, x_{100})$ be a permutation of $(1, 2, ..., 100)$. Define

$$S = \{m \mid m \text{ is the median of } \{x_i, x_{i+1}, x_{i+2}\} \text{ for some } i\}.$$

Determine the minimum possible value of the sum of all elements of S.

3 Let $ABCC_1B_1A_1$ be a convex hexagon such that $AB = BC$, and suppose that the line segments AA_1, BB_1, and CC_1 have the same perpendicular bisector. Let the diagonals AC_1 and A_1C meet at D, and denote by ω the circle ABC. Let ω intersect the circle A_1BC_1 again at $E \neq B$. Prove that the lines BB_1 and DE intersect on ω.

– Day 5

1 Let $ABCDE$ be a convex pentagon such that $AB = BC = CD$, $\angle EAB = \angle BCD$, and $\angle EDC = \angle CBA$. Prove that the perpendicular line from E to BC and the line segments AC and BD are concurrent.

2 Find all pairs (p, q) of prime numbers which $p > q$ and

$$\frac{(p+q)^{p+q}(p-q)^{p-q}-1}{(p+q)^{p-q}(p-q)^{p+q}-1}$$

is an integer.

3 Let $n > 1$ be a given integer. An $n \times n \times n$ cube is composed of n^3 unit cubes. Each unit cube is painted with one colour. For each $n \times n \times 1$ box consisting of n^2 unit cubes (in any of the three possible orientations), we consider the set of colours present in that box (each colour is listed only once). This way, we get $3n$ sets of colours, split into three groups according to the orientation.

It happens that for every set in any group, the same set appears in both of the other groups. Determine, in terms of n, the maximal possible number of colours that are present.

– Day 6

1 Let E and F be points on side BC of a triangle $\triangle ABC$. Points K and L are chosen on segments AB and AC, respectively, so that $EK \parallel AC$ and $FL \parallel AB$. The incircles of $\triangle BEK$ and $\triangle CFL$ touches segments AB and AC at X and Y, respectively. Lines AC and EX intersect at M, and lines AB and FY intersect at N. Given that $AX = AY$, prove that $MN \parallel BC$.

2 Call a rational number *short* if it has finitely many digits in its decimal expansion. For a positive integer m, we say that a positive integer t is $m-$*tastic* if there exists a number $c \in \{1, 2, 3, \ldots, 2017\}$ such that $\dfrac{10^t - 1}{c \cdot m}$ is short, and such that $\dfrac{10^k - 1}{c \cdot m}$ is not short for any $1 \leq k < t$. Let $S(m)$ be the set of $m-$tastic numbers. Consider $S(m)$ for $m = 1, 2, \ldots$. What is the maximum number of elements in $S(m)$?

3 Let $n \geq 3$ be an integer. Let $a_1, a_2, \ldots, a_n \in [0, 1]$ satisfy $a_1 + a_2 + \cdots + a_n = 2$. Prove that

$$\sqrt{1-\sqrt{a_1}} + \sqrt{1-\sqrt{a_2}} + \cdots + \sqrt{1-\sqrt{a_n}} \leq n - 3 + \sqrt{9 - 3\sqrt{6}}.$$

– Day 7

1 Find all functions $g : R \to R$ for which there exists a strictly increasing function $f : R \to R$ such that $f(x + y) = f(x)g(y) + f(y)\ \forall x, y \in R$.

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2 Sir Alex plays the following game on a row of 9 cells. Initially, all cells are empty. In each move, Sir Alex is allowed to perform exactly one of the following two operations:

- Choose any number of the form 2^j, where j is a non-negative integer, and put it into an empty cell.
- Choose two (not necessarily adjacent) cells with the same number in them; denote that number by 2^j. Replace the number in one of the cells with 2^{j+1} and erase the number in the other cell.

At the end of the game, one cell contains 2^n, where n is a given positive integer, while the other cells are empty. Determine the maximum number of moves that Sir Alex could have made, in terms of n.

Proposed by Warut Suksompong, Thailand

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3 Let n be a fixed odd positive integer. For each odd prime p, define

$$a_p = \frac{1}{p-1} \sum_{k=1}^{\frac{p-1}{2}} \left\{ \frac{k^{2n}}{p} \right\}.$$

Prove that there is a real number c such that $a_p = c$ for infinitely many primes p.

[i]Note: $\{x\} = x - \lfloor x \rfloor$ is the fractional part of x.[/i]

– Day 8

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1 Let n be a positive integer. Define a chameleon to be any sequence of $3n$ letters, with exactly n occurrences of each of the letters $a, b,$ and c. Define a swap to be the transposition of two adjacent letters in a chameleon. Prove that for any chameleon X , there exists a chameleon Y such that X cannot be changed to Y using fewer than $3n^2/2$ swaps.

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2 A sequence of real numbers $a_1, a_2, \dots$ satisfies the relation

$$a_n = -\max_{i+j=n} (a_i + a_j) \qquad \text{for all} \quad n > 2017.$$

Prove that the sequence is bounded, i.e., there is a constant M such that $|a_n| \leq M$ for all positive integers n.

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3 A convex quadrilateral $ABCD$ has an inscribed circle with center I. Let I_a, I_b, I_c and I_d be the incenters of the triangles DAB, ABC, BCD and CDA, respectively. Suppose that the common external tangents of the circles AI_bI_d and CI_bI_d meet at X, and the common external tangents of the circles BI_aI_c and DI_aI_c meet at Y. Prove that $\angle XIY = 90^\circ$.

– Day 9

1 Let $p \geq 2$ be a prime number. Eduardo and Fernando play the following game making moves alternately: in each move, the current player chooses an index i in the set $\{0, 1, 2, \ldots, p-1\}$ that was not chosen before by either of the two players and then chooses an element a_i from the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Eduardo has the first move. The game ends after all the indices have been chosen .Then the following number is computed:

$$M = a_0 + a_1 10 + a_2 10^2 + \cdots + a_{p-1}10^{p-1} = \sum_{i=0}^{p-1} a_i.10^i$$

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The goal of Eduardo is to make M divisible by p, and the goal of Fernando is to prevent this.

Prove that Eduardo has a winning strategy.

Proposed by Amine Natik, Morocco

2 In triangle ABC, let ω be the excircle opposite to A. Let D, E and F be the points where ω is tangent to BC, CA, and AB, respectively. The circle AEF intersects line BC at P and Q. Let M be the midpoint of AD. Prove that the circle MPQ is tangent to ω.

3 An integer $n \geq 3$ is given. We call an n-tuple of real numbers $(x_1, x_2, \ldots, x_n)$ *Shiny* if for each permutation $y_1, y_2, \ldots, y_n$ of these numbers, we have

$$\sum_{i=1}^{n-1} y_i y_{i+1} = y_1y_2 + y_2y_3 + y_3y_4 + \cdots + y_{n-1}y_n \geq -1.$$

Find the largest constant $K = K(n)$ such that

$$\sum_{1 \leq i < j \leq n} x_i x_j \geq K$$

holds for every Shiny n-tuple $(x_1, x_2, \ldots, x_n)$.