Caltech Harvey Mudd Math Competition from Fall 2016

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– Team Round

1 Let a_n be the nth positive integer such that when n is written in base 3, the sum of the digits of n is divisible by 3. For example, $a_1 = 5$ because $5 = 12_3$. Compute a_{2016}.

2 Consider the 5×5 grid $Z_5^2 = \{(a, b) : 0 \le a, b \le 4\}$.
Say that two points (a, b),(x, y) are adjacent if $a - x \equiv -1, 0, 1$ (mod 5) and $b - y \equiv -1, 0, 1$ (mod 5) .
For example, in the diagram, all of the squares marked with $\cdot$ are adjacent to the square marked with $\times$.
https://cdn.artofproblemsolving.com/attachments/2/6/c49dd26ab48ddff5e1beecfbd167d5bb9fe16
png
What is the largest number of $\times$ that can be placed on the grid such that no two are adjacent?

3 For a positive integer m, let $f(m)$ be the number of positive integers $q \le m$ such that $\frac{q^2-4}{m}$ is an integer. How many positive square-free integers $m < 2016$ satisfy $f(m) \ge 16$?

4 Line segments m and n both have length 2 and bisect each other at an angle of 60^o, as shown. A point X is placed at uniform random position along n, and a point Y is placed at a uniform random position along m. Find the probability that the distance between X and Y is less than $\frac{1}{2}$.

5 Given a triangle ABC, let D be a point on segment BC. Construct the circumcircle ω of triangle ABD and point E on ω such that CE is tangent to ω and A, E are on opposite sides of BC (as shown in diagram). If $\angle CAD = \angle ECD$ and $AC = 12$, $AB = 7$, find AE.

6 For any nonempty set of integers X, define the function

$$f(X) = \frac{(-1)^{|X|}}{\left(\prod_{k\in X} k\right)^2}$$

where $|X|$ denotes the number of elements in X.
Consider the set $S = \{2, 3, ..., 13\}$. Note that 1 is not an element of S.
Compute

$$\sum_{T\subseteq S, T\neq\emptyset} f(T).$$

where the sum is taken over all nonempty subsets T of S.

7 Consider constructing a tower of tables of numbers as follows. The first table is a one by one array containing the single number 1.
The second table is a two by two array formed underneath the first table and built as followed. For each entry, we look at the terms in the previous table that are directly up and to the left, up and to the right, and down and to the right of the entry, and we fill that entry with the sum of the numbers occurring there. If there happens to be no term at a particular location, it contributes a value of zero to the sum.
`https://cdn.artofproblemsolving.com/attachments/d/8/ab56dddfc23e84348e205f031001d157cb838`
`png`
The diagram above shows how we compute the second table from the first.
The diagram below shows how to then compute the third table from the second.
`https://cdn.artofproblemsolving.com/attachments/9/3/e1d8cf0fd0b71b970625a4fa97bc2912492a7`
`png`
For example, the entry in the middle row and middle column of the third table is equal the sum of the top left entry 1, the top right entry 0, and the bottom right entry 1 from the second table, which is just 2.
Similarly, to compute the bottom rightmost entry in the third table, we look above it to the left and see that the entry in the second table's bottom rightmost entry is 1. There are no entries from the second table above it and to the right or below it and to the right, so we just take this entry in the third table to be 1.
We continue constructing the tower by making more tables from the previous tables. Find the entry in the third (from the bottom) row of the third (from the left) column of the tenth table in this resulting tower.

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8 Let n be a positive integer. If S is a nonempty set of positive integers, then we say S is n-*complete* if all elements of S are divisors of n, and if d_1 and d_2 are any elements of S, then $n|d_1$ and gcd (d_1, d_2) are in S. How many 2310-complete sets are there?

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9 Find the sum of all 3-digit numbers whose digits, when read from left to right, form a strictly increasing sequence. (Numbers with a leading zero, e.g. "087" or "002", are not counted as having 3 digits.)

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10 Let ABC be a triangle with circumcircle ω such that $AB = 11$, $AC = 13$, and $\angle A = 30^o$. Points D and E are on segments AB and AC respectively such that $AD = 7$ and $AE = 8$. There exists a unique point $F \neq A$ on minor arc AB of ω such that $\angle FDA = \angle FEA$. Compute FA^2.

– Individual Round

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1 We say that the string $d_kd_{k-1}\cdots d_1d_0$ represents a number n in base -2 if each d_i is either 0 or 1,
and $n = d_k(-2)^k + d_{k-1}(-2)^{k-1} + \cdots + d_1(-2) + d_0$. For example, 110_{-2} represents the number 2. What string represents 2016 in base -2?

2 Alice and Bob find themselves on a coordinate plane at time $t = 0$ at $A(1,0)$ and $B(-1,0)$ respectively. They have no sense of direction, but they want to find each other. They each pick a direction independently and with uniform random probability. Both Alice and Bob travel at a constant speed of $1\frac{unit}{min}$ in their chosen directions. They continue on their straight line paths forever, each hoping to catch sight of the other. They both have a 1 unit radius of view; they can see something if and only if its distance from them is at most 1 unit. What is the probability they never see each other?

3 Two towns, A and B, are 100 miles apart. Every 20 minutes, (starting at midnight), a bus traveling at 60 mph leaves town A for town B, and every 30 minutes (starting at midnight) a bus traveling at 20 mph leaves town B for town A. Dirk starts in Town A and gets on a bus leaving for town B at noon. However, Dirk is always afraid he has boarded a bus going in the wrong direction, so each time the bus he is in passes another bus, he gets out and transfers to that other bus. How many hours pass before Dirk finally reaches Town B?

4 Compute

$$\sum_{n=1}^{\infty} \frac{2^{n+1}}{8 \cdot 4^n - 6 \cdot 2^n + 1}$$

5 Suppose you have 27 identical unit cubes colored such that 3 faces adjacent to a vertex are red and the other 3 are colored blue. Suppose further that you assemble these 27 cubes randomly into a larger cube with 3 cubes to an edge (in particular, the orientation of each cube is random). The probability that the entire cube is one solid color can be written as $\frac{1}{2^n}$ for some positive integer n. Find n.

6 How many binary strings of length 10 do not contain the substrings 101 or 010?

7 Let $f(x) = \frac{1}{1-\frac{3x}{16}}$. Consider the sequence $\{0, f(0), f(f(0)), f^3(0), \dots\}$ Find the smallest L such that $f^n(0) \le L$ for all n. If the sequence is unbounded, write none as your answer.

8 For positive integers n, d, define $n\%d$ to be the unique value of the positive integer $r < d$ such that $n = qd + r$, for some positive integer q. What is the smallest value of n not divisible by $5, 7, 11, 13$ for which $n^2\%5 < n^2\%7 < n^2\%11 < n^2\%13$?

9 In quadrilateral $ABCD$, $AB = DB$ and $AD = BC$. If $\angle ABD = 36^\circ$ and $\angle BCD = 54^\circ$, find $\angle ADC$ in degrees.

10 For a positive integer n, let $p(n)$ denote the number of prime divisors of n, counting multiplicity (i.e. $p(12) = 3$). A sequence a_n is defined such that $a_0 = 2$ and for $n > 0$, $a_n = 8^{p(a_{n-1})} + 2$.

Compute

$$\sum_{n=0}^{\infty} \frac{a_n}{2^n}$$

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11 Let $a, b \in [0,1], c \in [-1,1]$ be reals chosen independently and uniformly at random. What is the probability that $p(x) = ax^2 + bx + c$ has a root in $[0,1]$?

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12 For a positive real number a, let C be the cube with vertices at $(\pm a, \pm a, \pm a)$ and let T be the tetrahedron with vertices at $(2a, 2a, 2a), (2a, -2a, -2a), (-2a, 2a, -2a), (-2a, -2a, -2a)$. If the intersection of T and C has volume ka^3 for some k, find k.

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13 A sequence of numbers $a_1, a_2, \dots a_m$ is a *geometric sequence modulo n of length m* for $n, m \in \mathbb{Z}^+$ if for every index i, $a_i \in \{0, 1, 2, \dots, m-1\}$ and there exists an integer k such that $n | a_{j+1} - ka_j$ for $1 \leq j \leq m-1$. How many geometric sequences modulo 14 of length 14 are there?

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14 For a unit circle O, arrange points A, B, C, D and E in that order evenly along O's circumference. For each of those points, draw the arc centered at that point inside O from the point to its left to the point to its right. Denote the outermost intersections of these arcs as A', B', C', D' and E', where the prime of any point is opposite the point. The length of AC' can be written as an expression $f(x)$, where f is a trigonometric function. Find this expression.

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15 In a 5×5 grid of squares, how many nonintersecting pairs rectangles of rectangles are there? (Note sharing a vertex or edge still means the rectangles intersect.)
