Caltech Harvey Mudd Math Competition from Fall 2018

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by parmenides51

– Team Round

1 Anita plays the following single-player game: She is given a circle in the plane. The center of this circle and some point on the circle are designated ”known points”. Now she makes a series of moves, each of which takes one of the following forms:
(i) She draws a line (infinite in both directions) between two ”known points”; or
(ii) She draws a circle whose center is a ”known point” and which intersects another ”known point”.
Once she makes a move, all intersections between her new line/circle and existing lines/circles become ”known points”, unless the new/line circle is identical to an existing one. In other words, Anita is making a ruler-and-compass construction, starting from a circle.
What is the smallest number of moves that Anita can use to construct a drawing containing an equilateral triangle inscribed in the original circle?

2 Compute the sum $\sum_{n=1}^{200} \frac{1}{n(n+1)(n+2)}$.

3 Let p be the third-smallest prime number greater than 5 such that: • $2p+1$ is prime, and • $5^p \not\equiv 1 \pmod{2p+1}$.
Find p.

4 If Percy rolls a fair six-sided die until he rolls a 5, what is his expected number of rolls, given that all of his rolls are prime?

5 Let $\triangle ABC$ be a right triangle such that $AB = 3$, $BC = 4$, $AC = 5$. Let point D be on AC such that the incircles of $\triangle ABD$ and $\triangle BCD$ are mutually tangent. Find the length of BD.

6 Karina has a polynomial $p_1(x) = x^2 + x + k$, where k is an integer. Noticing that p_1 has integer roots, she forms a new polynomial $p_2(x) = x^2 + a_1x + b_1$, where a_1 and b_1 are the roots of p_1 and $a_1 \ge b_1$. The polynomial p_2 also has integer roots, so she forms a new polynomial $p_3(x) = x^2 + a_2x + b_2$, where a_2 and b_2 are the roots of p_2 and $a_2 \ge b_2$. She continues this process until she reaches $p_7(x)$ and finds that it does not have integer roots. What is the largest possible value of k?

7 For a positive number n, let $g(n)$ be the product of all $1 \le k \le n$ such that gcd $(k, n) = 1$, and say that $n > 1$ is reckless if n is odd and $g(n) \equiv -1 \pmod{n}$. Find the number of reckless numbers less than 50.

8 Find the largest positive integer n that cannot be written as $n = 20a + 28b + 35c$ for nonnegative integers a, b, and c.

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9 Say that a function $f : \{1, 2, ..., 1001\} \to Z$ is *almost* polynomial if there is a polynomial $p(x) = a_{200}x^{200} + ... + a_1x + a_0$ such that each an is an integer with $|a_n| \le 201$, and such that $|f(x) - p(x)| \le 1$ for all $x \in \{1, 2, ..., 1001\}$. Let N be the number of almost polynomial functions. Compute the remainder upon dividing N by 199.

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10 Let ABC be a triangle such that $AB = 13$, $BC = 14$, $AC = 15$. Let M be the midpoint of BC and define $P \neq B$ to be a point on the circumcircle of ABC such that $BP \perp PM$. Furthermore, let H be the orthocenter of ABM and define Q to be the intersection of BP and AC. If R is a point on HQ such that $RB \perp BC$, find the length of RB.

– Tiebreaker Round

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1 A large pond contains infinitely many lily pads labelled 1, 2, 3,..., placed in a line, where for each k, lily pad $k + 1$ is one unit to the right of lily pad k. A frog starts at lily pad 100. Each minute, if the frog is at lily pad n, it hops to lily pad $n + 1$ with probability $\frac{n-1}{n}$, and hops all the way back to lily pad 1 with probability $\frac{1}{n}$. Let N be the position of the frog after 1000 minutes. What is the expected value of N?

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2 A cat is tied to one corner of the base of a tower. The base forms an equilateral triangle of side length 4 m, and the cat is tied with a leash of length 8 m. Let A be the area of the region accessible to the cat. If we write $A = \frac{m}{n}k - \sqrt{\ell}$, where m, n, k, ℓ are positive integers such that m and n are relatively prime, and ℓ is squarefree, what is the value of $m + n + k + \ell$?

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3 Compute

$$\sum_{n=1}^{\infty}\left(\frac{1}{n^2 + 3n} - \frac{1}{n^2 + 3n + 2}\right)$$

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4 Find the sum of the real roots of $f(x) = x^4 + 9x^3 + 18x^2 + 18x + 4$.

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5 Let a, b, c, d, e be the roots of $p(x) = 2x^5 - 3x^3 + 2x - 7$. Find the value of

$$(a^3 - 1)(b^3 - 1)(c^3 - 1)(d^3 - 1)(e^3 - 1).$$

– Individual Round

Individual **p1.** Two robots race on the plane from $(0, 0)$ to (a, b), where a and b are positive real numbers with $a < b$. The robots move at the same constant speed. However, the first robot can only travel in directions parallel to the lines $x = 0$ or $y = 0$, while the second robot can only travel in directions parallel to the lines $y = x$ or $y = -x$. Both robots take the shortest possible path to (a, b) and arrive at the same time. Find the ratio $\frac{a}{b}$.

p2. Suppose $x + \frac{1}{x} + y + \frac{1}{y} = 12$ and $x^2 + \frac{1}{x^2} + y^2 + \frac{1}{y^2} = 70$. Compute $x^3 + \frac{1}{x^3} + y^3 + \frac{1}{y^3}$.

p3. Find the largest non-negative integer a such that 2^a divides

$$3^{2^{2018}} + 3.$$

p4. Suppose z and w are complex numbers, and $|z| = |w| = z\overline{w} + \overline{z}w = 1$. Find the largest possible value of $Re(z + w)$, the real part of $z + w$.

p5. Two people, A and B, are playing a game with three piles of matches. In this game, a move consists of a player taking a positive number of matches from one of the three piles such that the number remaining in the pile is equal to the nonnegative difference of the numbers of matches in the other two piles. A and B each take turns making moves, with A making the first move. The last player able to make a move wins. Suppose that the three piles have 10, x, and 30 matches. Find the largest value of x for which A does not have a winning strategy.

p6. Let $A_1A_2A_3A_4A_5A_6$ be a regular hexagon with side length 1. For $n = 1$,..., 6, let B_n be a point on the segment A_nA_{n+1} chosen at random (where indices are taken mod 6, so $A_7 = A_1$). Find the expected area of the hexagon $B_1B_2B_3B_4B_5B_6$.

p7. A termite sits at the point $(0, 0, 0)$, at the center of the octahedron $|x| + |y| + |z| \le 5$. The termite can only move a unit distance in either direction parallel to one of the x, y, or z axes: each step it takes moves it to an adjacent lattice point. How many distinct paths, consisting of 5 steps, can the termite use to reach the surface of the octahedron?

p8. Let

$$P(x) = x^{4037} - 3 - 8 \cdot \sum_{n=1}^{2018} 3^{n-1}x^n$$

Find the number of roots z of $P(x)$ with $|z| > 1$, counting multiplicity.

p9. How many times does 01101 appear as a not necessarily contiguous substring of 0101010101010101? (Stated another way, how many ways can we choose digits from the second string, such that when read in order, these digits read 01101?)

p10. A perfect number is a positive integer that is equal to the sum of its proper positive divisors, that is, the sum of its positive divisors excluding the number itself. For example, 28 is a perfect number because $1 + 2 + 4 + 7 + 14 = 28$. Let n_i denote the ith smallest perfect number. Define

$$f(x) = \sum_{i|n_x} \sum_{j|n_i} \frac{1}{j}$$

(where $\sum_{i|n_x}$ means we sum over all positive integers i that are divisors of n_x). Compute $f(2)$, given there are at least 50perfect numbers.

p11. Let O be a circle with chord AB. The perpendicular bisector to AB is drawn, intersecting O at points C and D, and intersecting AB at the midpoint E. Finally, a circle O' with diameter ED is drawn, and intersects the chord AD at the point F. Given $EC = 12$, and $EF = 7$, compute the radius of O.

p12. Suppose r, s, t are the roots of the polynomial $x^3 - 2x + 3$. Find

$$\frac{1}{r^3 - 2} + \frac{1}{s^3 - 2} + \frac{1}{t^3 - 2}.$$

p13. Let a_1, a_2,..., a_{14} be points chosen independently at random from the interval $[0, 1]$. For $k = 1$, 2,..., 7, let I_k be the closed interval lying between a_{2k-1} and a_{2k} (from the smaller to the larger). What is the probability that the intersection of I_1, I_2,..., I_7 is nonempty?

p14. Consider all triangles $\triangle ABC$ with area $144\sqrt{3}$ such that

$$\frac{\sin A \sin B \sin C}{\sin A + \sin B + \sin C} = \frac{1}{4}.$$

Over all such triangles ABC, what is the smallest possible perimeter?

p15. Let N be the number of sequences $(x_1, x_2, ..., x_{2018})$ of elements of $\{1, 2, ..., 2019\}$, not necessarily distinct, such that $x_1 + x_2 + ... + x_{2018}$ is divisible by 2018. Find the last three digits of N.

PS. You had better use hide for answers. Collected here (`https://artofproblemsolving.com/community/c5h2760506p24143309`).