Romania Team Selection Tests 2022

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by oVlad, VicKmath7, Lukaluce

Day 1 April 19

1 Given are positive reals $x_1, x_2, ..., x_n$ such that $\sum \frac{1}{1+x_i^2} = 1$. Find the minimal value of the expression $\frac{\sum x_i}{\sum \frac{1}{x_i}}$ and find when it is achieved.

2 Let $n \geq 2$ be an integer and let

$$M = \left\{ \frac{a_1 + a_2 + ... + a_k}{k} : 1 \leq k \leq n \text{ and } 1 \leq a_1 < \ldots < a_k \leq n \right\}$$

be the set of the arithmetic means of the elements of all non-empty subsets of $\{1, 2, ..., n\}$. Find

$$\min\{|a - b| : a, b \in M \text{ with } a \neq b\}.$$

3 Let ABC be an acute triangle such that $AB < AC$. Let ω be the circumcircle of ABC and assume that the tangent to ω at A intersects the line BC at D. Let Ω be the circle with center D and radius AD. Denote by E the second intersection point of ω and Ω. Let M be the midpoint of BC. If the line BE meets Ω again at X, and the line CX meets Ω for the second time at Y, show that A, Y, and M are collinear.

Proposed by Nikola Velov, North Macedonia

4 Can every positive rational number q be written as

$$\frac{a^{2021} + b^{2023}}{c^{2022} + d^{2024}},$$

where a, b, c, d are all positive integers?

Proposed by Dominic Yeo, UK

5 Given is an integer $k \geq 2$. Determine the smallest positive integer n, such that, among any n points in the plane, there exist k points among them, such that all distances between them are less than or equal to 2, or all distances are strictly greater than 1.

Day 2 May 14

1 Let ABC be an acute scalene triangle and let ω be its Euler circle. The tangent t_A of ω at the foot of the height A of the triangle ABC, intersects the circle of diameter AB at the point K_A for the second time. The line determined by the feet of the heights A and C of the triangle ABC intersects the lines AK_A and BK_A at the points L_A and M_A, respectively, and the lines t_A and CM_A intersect at the point N_A.

Points K_B, L_B, M_B, N_B and K_C, L_C, M_C, N_C are defined similarly for (B, C, A) and (C, A, B) respectively. Show that the lines $L_AN_A, L_BN_B,$ and L_CN_C are concurrent.

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2 Let ABC be an acute triangle and let B' and C' be the feet of the heights B and C of triangle ABC respectively. Let B'_A and B'_C be reflections of B' with respect to the lines BC and AB, respectively. The circle $BB'_AB'_C$, centered in O_B, intersects the line AB in X_B for the second time.

The points C'_A, C'_B, O_C, X_C are defined analogously, by replacing the pair (B, B') with the pair (C, C'). Show that O_BX_B and O_CX_C are parallel.

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3 Let $n \geq 2$ be an integer. Let $a_{ij},\ i,j = 1,2,\dots,n$ be n^2 positive real numbers satisfying the following conditions:

-For all $i = 1,\dots,n$ we have $a_{ii} = 1$ and,
-For all $j = 2,\dots,n$ the numbers $a_{ij},\ i = 1,\dots,j-1$ form a permutation of $1/a_{ji},\ i = 1,\dots,j-1.$

Given that $S_i = a_{i1} + \dots + a_{in}$, determine the maximum value of the sum $1/S_1 + \dots + 1/S_n$.

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4 Any positive integer N which can be expressed as the sum of three squares can obviously be written as

$$N = \frac{a^2 + b^2 + c^2 + d^2}{1 + abcd}$$

where a, b, c, d are nonnegative integers. Is the mutual assertion true?

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5 Let $m, n \geq 2$ be positive integers and $S \subseteq [1, m] \times [1, n]$ be a set of lattice points. Prove that if

$$|S| \geq m + n + \left\lfloor \frac{m+n}{4} - \frac{1}{2} \right\rfloor$$

then there exists a circle which passes through at least four distinct points of S.

Day 3 May 28

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1 Alice is drawing a shape on a piece of paper. She starts by placing her pencil at the origin, and then draws line segments of length one, alternating between vertical and horizontal segments. Eventually, her pencil returns to the origin, forming a closed, non-self-intersecting shape. Show that the area of this shape is even if and only if its perimeter is a multiple of eight.

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2 Let ABC be a triangle with $AB < AC$ and let D be the other intersection point of the angle bisector of $\angle A$ with the circumcircle of the triangle ABC. Let E and F be points on the sides AB and AC respectively, such that $AE = AF$ and let P be the point of intersection of AD and EF. Let M be the midpoint of BC. Prove that AM and the circumcircles of the triangles AEF and PMD pass through a common point.

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3 Consider a prime number $p \geqslant 11$. We call a triple a, b, c of natural numbers *suitable* if they give non-zero, pairwise distinct residues modulo p. Further, for any natural numbers a, b, c, k we define

$$f_k(a,b,c) = a(b-c)^{p-k} + b(c-a)^{p-k} + c(a-b)^{p-k}.$$

Prove that there exist suitable a, b, c for which $p \mid f_2(a,b,c)$. Furthermore, for each such triple, prove that there exists $k \geqslant 3$ for which $p \nmid f_k(a,b,c)$ and determine the minimal k with this property.

Călin Popescu and Marian Andronache

Day 4 May 29

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1 A finite set $\mathcal{L}$ of coplanar lines, no three of which are concurrent, is called *odd* if, for every line ℓ in $\mathcal{L}$ the total number of lines in $\mathcal{L}$ crossed by ℓ is odd.

-Prove that every finite set of coplanar lines, no three of which are concurrent, extends to an odd set of coplanar lines.
-Given a positive integer n determine the smallest nonnegative integer k satisfying the following condition: Every set of n coplanar lines, no three of which are concurrent, extends to an odd set of $n+k$ coplanar lines.

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2 Fix a nonnegative integer a_0 to define a sequence of integers $a_0, a_1, \ldots$ by letting $a_k, k \geq 1$ be the smallest integer (strictly) greater than a_{k-1} making $a_{k-1} + a_k$ into a perfect square. Let S be the set of positive integers not expressible as the difference of two terms of the sequence $(a_k)_{k\geq 0}$. Prove that S is finite and determine its size in terms of a_0.

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3 Let ABC be a triangle and let its incircle γ touch the sides BC, CA, AB at D, E, F respectively. Let P be a point strictly in the interior of γ. The segments PA, PB, PC cross γ at A_0, B_0, C_0 respectively. Let S_A, S_B, S_C be the centres of the circles PEF, PFD, PDE respectively and let T_A, T_B, T_C be the centres of the circles $PB_0C_0, PC_0A_0, PA_0B_0$ respectively. Prove that S_AT_A, S_BT_B and S_CT_C are concurrent.
