Vietnam National Olympiad 2015

www.artofproblemsolving.com/community/c303083

by gavrilos, quangminhltv99, quykhtn-qa1, shinny98NT

– Day 1

---

1 Given a non negative real a and a sequence (u_n) defined by

$$\begin{cases} u_1 = 3 \\ u_{n+1} = \frac{u_n}{2} + \frac{n^2}{4n^2+a}\sqrt{u_n^2+3} \end{cases}$$

a) Prove that for $a = 0$, the sequence is convergent and find its limit.

b) For $a \in [0, 1]$, prove that the sequence if convergent.

---

2 If a, b, c are nonnegative real numbers, then

$$3(a^2+b^2+c^2) \geq (a+b+c)(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}) + (a-b)^2 + (b-c)^2 + (c-a)^2 \geq (a+b+c)^2.$$

---

3 Given $m \in \mathbb{Z}^+$. Find all natural numbers n that does not exceed 10^m satisfying the following conditions:
i) $3|n$.
ii) The digits of n in decimal representation are in the set $\{2, 0, 1, 5\}$.

---

4 Given a circumcircle (O) and two fixed points B, C on (O). BC is not the diameter of (O). A point A varies on (O) such that ABC is an acute triangle. E, F is the foot of the altitude from B, C respectively of ABC. (I) is a variable circumcircle going through E and F with center I.

a) Assume that (I) touches BC at D. Probe that $\frac{DB}{DC} = \sqrt{\frac{\cot B}{\cot C}}$.

b) Assume (I) intersects BC at M and N. Let H be the orthocenter and P, Q be the intersections of (I) and (HBC). The circumcircle (K) going through P, Q and touches (O) at T (T is on the same side with A wrt PQ). Prove that the interior angle bisector of $\angle MTN$ passes through a fixed point.

– Day 2

---

1 Let $\{f(x)\}$ be a sequence of polynomial, where $f_0(x) = 2$, $f_1(x) = 3x$, and

$f_n(x) = 3xf_{n-1}(x) + (1 - x - 2x^2)f_{n-2}(x)$ $(n \geq 2)$

Determine the value of n such that $f_n(x)$ is divisible by $x^3 - x^2 + x$.

---

2 For $a, n \in \mathbb{Z}^+$, consider the following equation:

$$a^2x + 6ay + 36z = n \quad (1)$$

where $x, y, z \in \mathbb{N}$.

a) Find all a such that for all $n \geq 250$, (1) always has natural roots (x, y, z).

b) Given that $a > 1$ and $\gcd(a, 6) = 1$. Find the greatest value of n in terms of a such that (1) doesn't have natural root (x, y, z).

---

3 There are m boys and n girls participate in a duet singing contest ($m, n \geq 2$). At the contest, each section there will be one show. And a show including some boy-girl duets where each boy-girl couple will sing with each other no more than one song and each participant will sing at least one song. Two show A and B are considered different if there is a boy-girl couple sing at show A but not show B. The contest will end if and only if every possible shows are performed, and each show is performed exactly one time.

a) A show is called *depend* on a participant X if we cancel all duets that X perform, then there will be at least one another participant will not sing any song in that show. Prove that among every shows that depend on X, the number of shows with odd number of songs equal to the number of shows with even number of songs.

b) Prove that the organizers can arrange the shows in order to make sure that the numbers of songs in two consecutive shows have different parity.
