This is a collection of problems from Kazakstan national olympiad 2020, including grade 9-11 problems. Due to covid-19, the olympiad was held online in one-day 4 problems format instead of usual two-days 6 problems.

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– Grade 9

1 There are n lamps and k switches in a room. Initially, each lamp is either turned on or turned off. Each lamp is connected by a wire with 2020 switches. Switching a switch changes the state of a lamp, that is connected to it, to the opposite state. It is known that one can switch the switches so that all lamps will be turned on. Prove, that it is possible to achieve the same result by switching the switches no more than $\left\lfloor \frac{k}{2} \right\rfloor$ times.

Proposed by T. Zimanov

2 Let $x_1, x_2, ..., x_n$ be a real numbers such that

1) $1 \le x_1, x_2, ..., x_n \le 160$
2) $x_i^2 + x_j^2 + x_k^2 \ge 2(x_ix_j + x_jx_k + x_kx_i)$ for all $1 \le i < j < k \le n$
Find the largest possible n.

3 Let p be a prime number and k, r are positive integers such that $p > r$. If $pk + r$ divides $p^p + 1$ then prove that r divides k.

4 The incircle of the triangle ABC touches the sides of AB, BC, CA at points C_0, A_0, B_0, respectively. Let the point M be the midpoint of the segment connecting the vertex C_0 with the intersection point of the altitudes of the triangle $A_0B_0C_0$, point N be the midpoint of the arc ACB of the circumscribed circle of the triangle ABC. Prove that line MN passes through the center of incircle of triangle ABC.

– Grades 10-11

1 Find all pairs (m, n) of natural numbers such that $n^4 \mid 2m^5 - 1$ and $m^4 \mid 2n^5 + 1$.

2 Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that for any $x, y \in \mathbb{R}^+$ the following equality holds:

$$f(x)f(y) = f\left(\frac{xy}{xf(x) + y}\right).$$

$\mathbb{R}^+$ denotes the set of positive real numbers.

3 A point N is marked on the median CM of the triangle ABC so that $MN \cdot MC = AB^2/4$. Lines AN and BN intersect the circumcircle $\triangle ABC$ for the second time at points P and Q, respectively. R is the point of segment PQ, nearest to Q, such that $\angle NRC = \angle BNC$. S is the point of the segment PQ closest to P such that $\angle NSC = \angle ANC$. Prove that $RN = SN$.

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4 Alice and Bob play a game on the infinite side of a checkered strip, in which the cells are numbered with consecutive integers from left to right (..., -2, -1, 0, 1, 2, ...). Alice in her turn puts one cross in any free cell, and Bob in his turn puts zeros in any 2020 free cells. Alice will win if he manages to get such 4 cells marked with crosses, the corresponding cell numbers will form an arithmetic progression. Bob's goal in this game is to prevent Alice from winning. They take turns and Alice moves first. Will Alice be able to win no matter how Bob plays?
