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– Grade 9

1 CH is an altitude in a right triangle ABC $(\angle C = 90^\circ)$. Points P and Q lie on AC and BC respectively such that $HP \perp AC$ and $HQ \perp BC$. Let M be an arbitrary point on PQ. A line passing through M and perpendicular to MH intersects lines AC and BC at points R and S respectively. Let M_1 be another point on PQ distinct from M. Points R_1 and S_1 are determined similarly for M_1. Prove that the ratio $\frac{RR_1}{SS_1}$ is constant.

2 Given a prime number p. It is known that for each integer a such that $1 < a < p/2$ there exist integer b such that $p/2 < b < p$ and $p|ab-1$. Find all such p.

3 See problem 2 from grades 10-11

4 P and Q are points on angle bisectors of two adjacent angles. Let PA, PB, QC and QD be altitudes on the sides of these adjacent angles. Prove that lines AB, CD and PQ are concurrent.

5 For positive reals a, b, c with $\sqrt{a}+\sqrt{b}+\sqrt{c} \ge 3$ prove that

$$\frac{a^3}{a^2+b}+\frac{b^3}{b^2+c}+\frac{c^3}{c^2+a} \ge \frac{3}{2}$$

6 Numbers from 1 to 49 are randomly placed in a 35×35 table such that number i is used exactly i times. Some random cells of the table are removed so that table falls apart into several connected (by sides) polygons. Among them, the one with the largest area is chosen (if there are several of the same largest areas, a random one of them is chosen). What is the largest number of cells that can be removed that guarantees that in the chosen polygon there is a number which occurs at least 15 times?

– Grades 10-11

1 Given a triangle ABC draw the altitudes AD, BE, CF. Take points P and Q on AB and AC, respectively such that $PQ \parallel BC$. Draw the circles with diameters BQ and CP and let them intersect at points R
and T where R is closer to A than T. Draw the altitudes BN and CM in the triangle BCR. Prove that FM, EN and AD are concurrent.

2 We define the function $Z(A)$ where we write the digits of A in base 10 form in reverse. (For example: $Z(521) = 125$). Call a number A *good* if the first and last digits of A are different, none of it's digits are 0 and the equality:

$$Z(A^2) = (Z(A))^2$$

happens. Find all such good numbers greater than 10^6.

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3 Given $m \in \mathbb{N}$. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$f(f(x) + y) - f(x) = \left(\frac{f(y)}{y} - 1\right) x + f^m(y)$$

holds for all $x, y \in \mathbb{R}^+$.

($f^m(x) = f$ applies m times.)

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4 Let $P(x)$ be a polynomial with positive integer coefficients such that $deg(P) = 699$. Prove that if $P(1) \leq 2022$, then there exist some consecutive coefficients such that their sum is 22, 55, or 77.

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5 Given a cyclic quadrilateral $ABCD$, let it's diagonals intersect at the point O. Take the midpoints of AD and BC as M and N respectively. Take a point S on the arc AB not containing C or D such that

$$\angle SMA = \angle SNB$$

Prove that if the diagonals of the quadrilateral made from the lines SM, SN, AB, and CD intersect at the point T, then S, O, and T are collinear.

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6 Given an infinite positive integer sequence $\{x_i\}$ such that

$$x_{n+2} = x_n x_{n+1} + 1$$

Prove that for any positive integer i there exists a positive integer j such that x_j^j is divisible by x_i^i.

[i]Remark: Unfortunately, there was a mistake in the problem statement during the contest itself. In the last sentence, it should say ”for any positive integer $i > 1$ ...”[/i]
