**Final Round - 2021 Korea**

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**Day** 1

**P1** Let $ABC$ be an acute triangle with circumcenter $O$, and let $D$, $E$, and $F$ be the feet of altitudes from $A$, $B$, and $C$ to sides $BC$, $CA$, and $AB$, respectively. Denote by $P$ the intersection of the tangents to the circumcircle of $ABC$ at $B$ and $C$. The line through $P$ perpendicular to $EF$ meets $AD$ at $Q$, and let $R$ be the foot of the perpendicular from $A$ to $EF$. Prove that $DR$ and $OQ$ are parallel.

**P2** There are $n$ boxes $A_1, ..., A_n$ with non-negative number of pebbles inside it(so it can be empty). Let $a_n$ be the number of pebbles in the box $A_n$. There are total $3n$ pebbles in the boxes. From now on, Alice plays the following operation.

In each operation, Alice choose one of these boxes which is non-empty. Then she divide this pebbles into $n$ group such that difference of number of pebbles in any two group is at most 1, and put these $n$ group of pebbles into $n$ boxes one by one. This continues until only one box has all the pebbles, and the rest of them are empty. And when it's over, define $Length$ as the total number of operations done by Alice.

Let $f(a_1, ..., a_n)$ be the smallest value of $Length$ among all the possible operations on $(a_1, ..., a_n)$. Find the maximum possible value of $f(a_1, ..., a_n)$ among all the ordered pair $(a_1, ..., a_n)$, and find all the ordered pair $(a_1, ..., a_n)$ that equality holds.

**P3** A function $g\colon \mathbb{R} \to \mathbb{R}$ is given such that its range is a finite set. Find all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfies

$$2f(x + g(y)) = f(2g(x) + y) + f(x + 3g(y))$$

for all $x, y \in \mathbb{R}$.

**Day** 2

**P4** Let $ABC$ be a scalene triangle with incenter $I$ and let $AI$ meet the circumcircle of triangle $ABC$ again at $M$. The incircle $\omega$ of triangle $ABC$ is tangent to sides $AB, AC$ at $D, E$, respectively. Let $O$ be the circumcenter of triangle $BDE$ and let $L$ be the intersection of $\omega$ and the altitude from $A$ to $BC$ so that $A$ and $L$ lie on the same side with respect to $DE$. Denote by $\Omega$ a circle centered at $O$ and passing through $L$, and let $AL$ meet $\Omega$ again at $N$.

Prove that the lines $LD$ and $MB$ meet on the circumcircle of triangle $LNE$.

**P5** Find all positive integers $m$ such that there exists integers $x$ and $y$ that satisfies

$$m \mid x^2 + 11y^2 + 2022.$$

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**P6** Set $X$ is called *fancy* if it satisfies all of the following conditions:

-The number of elements of $X$ is $2022$.
-Each element of $X$ is a closed interval contained in $[0, 1]$.
-For any real number $r \in [0, 1]$, the number of elements of $X$ containing $r$ is less than or equal to $1011$.

For *fancy* sets $A, B$, and intervals $I \in A, J \in B$, denote by $n(A, B)$ the number of pairs $(I, J)$ such that $I \cap J \neq \emptyset$. Determine the maximum value of $n(A, B)$.

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