Vietnam Team Selection Test 2022

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–	Day 1
1	Given a real number α and consider function $\varphi(x) = x^2 e^{\alpha x}$ for $x \in \mathbb{R}$. Find all function $f : \mathbb{R} \to \mathbb{R}$ that satisfy: $$f(\varphi(x) + f(y)) = y + \varphi(f(x))$$ forall $x, y \in \mathbb{R}$
2	Given a convex polyhedron with 2022 faces. In 3 arbitary faces, there are already number $26; 4$ and 2022 (each face contains 1 number). They want to fill in each other face a real number that is an arithmetic mean of every numbers in faces that have a common edge with that face. Prove that there is only one way to fill all the numbers in that polyhedron.
3	Let $ABCD$ be a parallelogram, AC intersects BD at I. Consider point G inside $\triangle ABC$ that satisfy $\angle IAG = \angle IBG \neq 45^\circ - \frac{\angle AIB}{4}$. Let E, G be projections of C on AG and D on BG. The $E-$ median line of $\triangle BEF$ and $F-$ median line of $\triangle AEF$ intersects at H. $a)$ Prove that AF, BE and IH concurrent. Call the concurrent point L. $b)$ Let K be the intersection of CE and DF. Let J circumcenter of (LAB) and M, N are respectively be circumcenters of (EIJ) and (FIJ). Prove that EM, FN and the line go through circumcenters of $(GAB), (KCD)$ are concurrent.
–	Day 2
4	An acute, non-isosceles triangle ABC is inscribed in a circle with centre O. A line go through O and midpoint I of BC intersects AB, AC at E, F respectively. Let D, G be reflections to A over O and circumcentre of (AEF), respectively. Let K be the reflection of O over circumcentre of (OBC). $a)$ Prove that D, G, K are collinear. $b)$ Let M, N are points on KB, KC that $IM \perp AC$, $IN \perp AB$. The midperpendiculars of IK intersects MN at H. Assume that IH intersects AB, AC at P, Q respectively. Prove that the circumcircle of $\triangle APQ$ intersects (O) the second time at a point on AI.
5	A fractional number x is called ***pretty*** if it has finite expression in base$-b$ numeral system, b is a positive integer in $[2; 2022]$. Prove that there exists finite positive integers $n \geq 4$ that with every m in $(\frac{2n}{3}; n)$ then there is at least one pretty number between $\frac{m}{n-m}$ and $\frac{n-m}{m}$
6	Given a set $A = \{1; 2; ...; 4044\}$. They color 2022 numbers of them by white and the rest of them by black. With each $i \in A$, called the ***important number*** of i be the number of all white numbers smaller than i and black numbers larger than i. With every natural number m, find all positive integers k that exist a way to color the numbers that can get k important numbers equal to m.