The problems from the 29th Macedonian Mathematical Olympiad

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Problem 1 Let $(x_n)_{n=1}^{\infty}$ be a sequence defined recursively with: $x_1 = 2$ and $x_{n+1} = \frac{x_n(x_n+n)}{n+1}$ for all $n \geq 1$. Prove that

$$n(n+1) > \frac{(x_1 + x_2 + \ldots + x_n)^2}{x_{n+1}}.$$

Proposed by Nikola Velov

Problem 2 Let $ABCD$ be cyclic quadrilateral and E the midpoint of AC. The circumcircle of $\triangle CDE$ intersect the side BC at F, which is different from C. If B' is the reflection of B across F, prove that EF is tangent to the circumcircle of $\triangle B'DF$.

Proposed by Nikola Velov

Problem 3 The sequence $(a_n)_{n\geq 1}^{\infty}$ is given by: $a_1 = 2$ and $a_{n+1} = a_n^2 + a_n$ for all $n \geq 1$.
For an integer $m \geq 2$, $L(m)$ denotes the greatest prime divisor of m. Prove that there exists some k, for which $L(a_k) > 1000^{1000}$.

Proposed by Nikola Velov

Problem 4 Sofia and Viktor are playing the following game on a 2022×2022 board:
- Firstly, Sofia covers the table completely by dominoes, no two are overlapping and all are inside the table;
- Then Viktor without seeing the table, chooses a positive integer n;
- After that Viktor looks at the table covered with dominoes, chooses and fixes n of them;
- Finally, Sofia removes the remaining dominoes that aren't fixed and tries to recover the table with dominoes differently from before.
If she achieves that, she wins, otherwise Viktor wins. What is the minimum number n for which Viktor can always win, no matter the starting covering of dominoes.

Proposed by Viktor Simjanoski

Problem 5 An acute $\triangle ABC$ with circumcircle Γ is given. I and I_a are the incenter and $A-$excenter of $\triangle ABC$ respectively. The line AI intersects Γ again at D and A' is the antipode of A with respect to Γ. X and Y are point on Γ such that $\angle IXD = \angle I_aYD = 90^\circ$. The tangents to Γ at X and Y intersect in point Z. Prove that A', D and Z are collinear.

Proposed by Nikola Velov

