South East Mathematical Olympiad 2016

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– Grade 10

Day 1

1 The sequence (a_n) is defined by $a_1 = 1, a_2 = \frac{1}{2}$,

$$n(n+1)a_{n+1}a_n + na_na_{n-1} = (n+1)^2 a_{n+1}a_{n-1} (n \geq 2).$$

Prove that

$$\frac{2}{n+1} < \sqrt[n]{a_n} < \frac{1}{\sqrt{n}} (n \geq 3).$$

2 Suppose PAB and PCD are two secants of circle O. Lines $AD \cap BC = Q$. Point T lie on segment BQ and point K is intersection of segment PT with circle O, $S = QK \cap PA$
Given that $ST \parallel PQ$, prove that B, S, K, T lie on a circle.

3 Given any integer $n \geq 3$. A finite series is called n-series if it satisfies the following two conditions $1)$ It has at least 3 terms and each term of it belongs to $\{1, 2, ..., n\}$ $2)$ If series has m terms $a_1, a_2, ..., a_m$ then $(a_{k+1} - a_k)(a_{k+2} - a_k) < 0$ for all $k = 1, 2, ..., m-2$

How many n-series are there $?$

4 For any four points on a plane, if the areas of four triangles formed are different positive integer and six distances between those four points are also six different positive integers, then the convex closure of 4 points is called a "lotus design."
(1) Construct an example of "lotus design". Also what are areas and distances in your example?
(2) Prove that there are infinitely many "lotus design" which are not similar.

Day 2

5 Let n is positive integer, D_n is a set of all positive divisor of n and $f(n) = \sum_{d \in D_n} \frac{1}{1+d}$
Prove that for all positive integer m, $\sum_{i=1}^{m} f(i) < m$

6 Toss the coin n times, assume that each time, only appear only head or tail
Let $a(n)$ denote number of way that head appear in multiple of 3 times among n times
Let $b(n)$ denote numbe of way that head appear in multiple of 6 times among n times (1) Find $a(2016)$ and $b(2016)$ (2) Find the number of positive integer $n \leq 2016$ that $2b(n) - a(n) \geq 0$

7 I is incenter of $\triangle ABC$. The incircle touches BC, CA, AB at D, E, F, respectively .
Let $M, N, K = BI, CI, DI \cap EF$ respectively and $BN \cap CM = P, AK \cap BC = G$.
Point Q is intersection of the perpendicular line to PG through I and the perpendicular line to PB through P.
Prove that BI bisect segment PQ.

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8 Let $\{a_n\}$ be a series consisting of positive integers such that $n^2 \mid \sum_{i=1}^{n} a_i$ and $a_n \leq (n+2016)^2$ for all $n \geq 2016$.
Define $b_n = a_{n+1} - a_n$. Prove that the series $\{b_n\}$ is eventually constant.

– Grade 11

Day 1

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2 Let n be positive integer,$x_1, x_2, \cdots, x_n$ be positive real numbers such that $x_1 x_2 \cdots x_n = 1$. Prove that

$$\sum_{i=1}^{n} x_i \sqrt{x_1^2 + x_2^2 + \cdots x_i^2} \geq \frac{n+1}{2}\sqrt{n}$$

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4 A substitute teacher lead a groop of students to go for a trip. The teacher who in charge of the groop of the students told the substitude teacher that there are two students who always lie, and the others always tell the truth. But the substitude teacher don't know who are the two students always lie. They get lost in a forest. Finally the are in a crossroad which has four roads. The substitute teacher knows that their camp is on one road, and the distence is 20 minutes' walk. The students have to go to the camp before it gets dark. (1) If there are 8 students, and 60 minutes before it gets dark, give a plan that all students can get back to the camp. (2) If there are 4 students, and 100 minutes before it gets dark, is there a plan that all students can get back to the camp?

Day 2

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5 Let a constant α as $0 < \alpha < 1$, prove that: (1) There exist a constant $C(\alpha)$ which is only depend on α such that for every $x \geq 0, \ln(1+x) \leq C(\alpha)x^{\alpha}$. (2) For every two complex numbers z_1, z_2, $|\ln|\frac{z_1}{z_2}|| \leq C(\alpha)\left(|\frac{z_1-z_2}{z_2}|^{\alpha} + |\frac{z_2-z_1}{z_1}|^{\alpha}\right)$.

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7 Let $A = \{a^3 + b^3 + c^3 - 3abc | a, b, c \in \mathbb{N}\}$, $B = \{(a+b-c)(b+c-a)(c+a-b) | a, b, c \in \mathbb{N}\}$, $P = \{n | n \in A \cap B, 1 \leq n \leq 2016\}$, find the value of $|P|$.