Sharygin Geometry Olympiad 2016

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– Grade 8

– Day 1

1 An altitude AH of triangle ABC bisects a median BM. Prove that the medians of triangle ABM are sidelengths of a right-angled triangle.

by Yu.Blinkov

2 A circumcircle of triangle ABC meets the sides AD and CD of a parallelogram $ABCD$ at points K and L respectively. Let M be the midpoint of arc KL not containing B. Prove that $DM \perp AC$.

by E.Bakaev

3 A trapezoid $ABCD$ and a line ℓ perpendicular to its bases AD and BC are given. A point X moves along ℓ. The perpendiculars from A to BX and from D to CX meet at point Y . Find the locus of Y .

by D.Prokopenko

4 Is it possible to dissect a regular decagon along some of its diagonals so that the resulting parts can form two regular polygons?

by N.Beluhov

– Day 2

5 Three points are marked on the transparent sheet of paper. Prove that the sheet can be folded along some line in such a way that these points form an equilateral triangle.

by A.Khachaturyan

6 A triangle ABC with $\angle A = 60^o$ is given. Points M and N on AB and AC respectively are such that the circumcenter of ABC bisects segment MN. Find the ratio $AN : MB$.

by E.Bakaev

7 Diagonals of a quadrilateral $ABCD$ are equal and meet at point O. The perpendicular bisectors to segments AB and CD meet at point P, and the perpendicular bisectors to BC and AD meet at point Q. Find angle $\angle POQ$.

by A.Zaslavsky

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8 A criminal is at point X, and three policemen at points A, B and C block him up, i.e. the point X lies inside the triangle ABC. Each evening one of the policemen is replaced in the following way: a new policeman takes the position equidistant from three former policemen, after this one of the former policemen goes away so that three remaining policemen block up the criminal too. May the policemen after some time occupy again the points A, B and C (it is known that at any moment X does not lie on a side of the triangle)?

by V.Protasov

– Grade 9

– Day 1

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1 The diagonals of a parallelogram $ABCD$ meet at point O. The tangent to the circumcircle of triangle BOC at O meets ray CB at point F. The circumcircle of triangle FOD meets BC for the second time at point G. Prove that $AG = AB$.

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2 Let H be the orthocenter of an acute-angled triangle ABC. Point X_A lying on the tangent at H to the circumcircle of triangle BHC is such that $AH = AX_A$ and $X_A \neq H$. Points X_B, X_C are defined similarly. Prove that the triangle $X_AX_BX_C$ and the orthotriangle of ABC are similar.

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3 Let O and I be the circumcenter and incenter of triangle ABC. The perpendicular from I to OI meets AB and the external bisector of angle C at points X and Y respectively. In what ratio does I divide the segment XY?

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4 One hundred and one beetles are crawling in the plane. Some of the beetles are friends. Every one hundred beetles can position themselves so that two of them are friends if and only if they are at unit distance from each other. Is it always true that all one hundred and one beetles can do the same?

– Day 2

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5 The center of a circle ω_2 lies on a circle ω_1. Tangents XP and XQ to ω_2 from an arbitrary point X of ω_1 (P and Q are the touching points) meet ω_1 for the second time at points R and S. Prove that the line PQ bisects the segment RS.

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6 The sidelines AB and CD of a trapezoid meet at point P, and the diagonals of this trapezoid meet at point Q. Point M on the smallest base BC is such that $AM = MD$. Prove that

$\angle PMB = \angle QMB$.

7 From the altitudes of an acute-angled triangle, a triangle can be composed. Prove that a triangle can be composed from the bisectors of this triangle.

8 The diagonals of a cyclic quadrilateral meet at point M. A circle ω touches segments MA and MD at points P, Q respectively and touches the circumcircle of $ABCD$ at point X. Prove that X lies on the radical axis of circles ACQ and BDP.

(Proposed by Ivan Frolov)

– Grade 10

– Day 1

1 A line parallel to the side BC of a triangle ABC meets the sides AB and AC at points P and Q, respectively. A point M is chosen inside the triangle APQ. The segments MB and MC meet the segment PQ at points E and F, respectively. Let N be the second intersection point of the circumcircles of the triangles PMF and QME. Prove that the points A, M, N are collinear.

2 Let I and I_a be the incenter and excenter (opposite vertex A) of a triangle ABC, respectively. Let A' be the point on its circumcircle opposite to A, and A_1 be the foot of the altitude from A. Prove that $\angle IA_1I_a = \angle IA'I_a$.

(Proposed by Pavel Kozhevnikov)

3 Assume that the two triangles ABC and $A'B'C'$ have the common incircle and the common circumcircle. Let a point P lie inside both the triangles. Prove that the sum of the distances from P to the sidelines of triangle ABC is equal to the sum of distances from P to the sidelines of triangle $A'B'C'$.

4 The Devil and the Man play a game. Initially, the Man pays some cash s to the Devil. Then he lists some 97 triples $\{i, j, k\}$ consisting of positive integers not exceeding 100. After that, the Devil draws some convex polygon $A_1A_2...A_{100}$ with area 100 and pays to the Man, the sum of areas of all triangles $A_iA_jA_k$. Determine the maximal value of s which guarantees that the Man receives at least as much cash as he paid.

Proposed by Nikolai Beluhov, Bulgaria

– Day 2

5 Does there exist a convex polyhedron having equal number of edges and diagonals?

(A diagonal of a polyhedron is a segment through two vertices not lying on the same face)

6 A triangle ABC is given. The point K is the base of the external bisector of angle A. The point M is the midpoint of the arc AC of the circumcircle. The point N on the bisector of angle C is such that $AN \parallel BM$. Prove that the points M, N, K are collinear.

(Proposed by Ilya Bogdanov)

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7 Restore a triangle by one of its vertices, the circumcenter and the Lemoine's point.

(The Lemoine's point is the intersection point of the reflections of the medians in the correspondent angle bisectors)

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8 Let ABC be a non-isosceles triangle, let AA_1 be its angle bisector and A_2 be the touching point of the incircle with side BC. The points B_1, B_2, C_1, C_2 are defined similarly. Let O and I be the circumcenter and the incenter of triangle ABC. Prove that the radical center of the circumcircle of the triangles $AA_1A_2, BB_1B_2, CC_1C_2$ lies on the line OI.

– First Round

– Grade 8

P1 A trapezoid $ABCD$ with bases AD and BC is such that $AB = BD$. Let M be the midpoint of DC. Prove that $\angle MBC$ = $\angle BCA$.

P2 Mark three nodes on a cellular paper so that the semiperimeter of the obtained triangle would be equal to the sum of its two smallest medians.

(Proposed by L.Emelyanov)

P3 Let AH_1, BH_2 be two altitudes of an acute-angled triangle ABC , D be the projection of H_1 to AC, E be the projection of D to AB, F be the common point of ED and AH_1.
Prove that $H_2F \parallel BC$.

(Proposed by E.Diomidov)

P4 In quadrilateral $ABCD$, $\angle B = \angle D = 90$ and $AC = BC + DC$. Point P of ray BD is such that $BP = AD$. Prove that line CP is parallel to the bisector of angle ABD.

(Proposed by A.Trigub)

P5 In quadrilateral $ABCD$, $AB = CD$, M and K are the midpoints of BC and AD.Prove that the angle between MK and AC is equal to the half-sum of angles BAC and DCA

(Proposed by M.Volchkevich)

P6 Let M be the midpoint of side AC of triangle ABC, MD and ME be the perpendiculars from M to AB and BC respectively. Prove that the distance between the circumcenters of triangles ABE and BCD is equal to $AC/4$

(Proposed by M.Volchkevich)

– Grades 89

P7 Let all distances between the vertices of a convex n-gon ($n > 3$) be
different.
a) A vertex is called uninteresting if the closest vertex is adjacent to it. What is the
minimal possible number of uninteresting vertices (for a given n)?
b) A vertex is called unusual if the farthest vertex is adjacent to it. What is the maximal
possible number of unusual vertices (for a given n)?

(Proposed by B.Frenkin)

P8 Let $ABCDE$ be an inscribed pentagon such that $\angle B + \angle E = \angle C + \angle D$.Prove that $\angle CAD < \pi/3 < \angle A$.

(Proposed by B.Frenkin)

P9 Let ABC be a right-angled triangle and CH be the altitude from its right angle C. Points O_1 and O_2 are the incenters of triangles ACH and BCH respectively, P_1 and P_2 are the touching points of their incircles with AC and BC. Prove that lines O_1P_1 and O_2P_2 meet on AB.

P10 Point X moves along side AB of triangle ABC, and point Y moves along its circumcircle in such a way that line XY passes through the midpoint of arc AB. Find the locus of the circumcenters of triangles IXY , where I is the incenter of ABC.

– Grades 810

P11 Restore a triangle ABC by vertex B, the centroid and the common point of the symmedian from B with the circumcircle.

– Grades 910

P12 Let BB_1 be the symmedian of a nonisosceles acute-angled triangle ABC. Ray BB_1 meets the circumcircle of ABC for the second time at point L. Let AH_A, BH_B, CH_C be the altitudes of triangle ABC. Ray BH_B meets the circumcircle of ABC for the second time at point T. Prove that H_A, H_C, T, L are concyclic.

P13 L is a Line that intersect with the side AB, BC, AC of triangle ABC at F, D, E
The line perpendicular to BC from D intersect AB, AC at A_1, A_2 respectively

Name B_1, B_2, C_1, C_2 similarly
Prove that the circumcenters of $AA_1A_2, BB_1B_2, CC_1C_2$ are collinear

– Grade 911

P14 Let a triangle ABC be given. Consider the circle touching its circumcircle at A and touching externally its incircle at some point A_1. Points B_1 and C_1 are defined similarly.
a) Prove that lines AA_1, BB_1 and $CC1$ concur.
b) Let A_2 be the touching point of the incircle with BC. Prove that lines AA_1 and AA_2 are symmetric about the bisector of angle $\angle A$.

P15 Let O, M, N be the circumcenter, the centroid and the Nagel point of a triangle. Prove that angle MON is right if and only if one of the triangles angles is equal to 60^o.

P16 Let BB_1 and CC_1 be altitudes of triangle ABC. The tangents to the circumcircle of AB_1C_1 at B_1 and C_1 meet AB and AC at points M and N respectively. Prove that the common point of circles AMN and AB_1C_1 distinct from A lies on the Euler line of ABC.

P17 Let D be an arbitrary point on side BC of triangle ABC. Circles ω_1 and ω_2 pass through A and D in such a way that BA touches ω_1 and CA touches ω_2. Let BX be the second tangent from B to ω_1, and CY be the second tangent from C to ω_2. Prove that the circumcircle of triangle XDY touches BC.

P18 Let ABC be a triangle with $\angle C = 90^\circ$, and K, L be the midpoints of the minor arcs AC and BC of its circumcircle. Segment KL meets AC,at point N. Find angle NIC where $I$is the incenter of ABC.

P19 Let $ABCDEF$ be a regular hexagon. Points P and Q on tangents to its circumcircle at A and D respectively are such that PQ touches the minor arc EF of this circle. Find the angle between PB and QC.

– Grades 1011

P20 The incircle ω of a triangle ABC touches BC, AC and AB at points A_0, B_0 and C_0 respectively. The bisectors of angles B and C meet the perpendicular bisector to segment AA_0 at points Q and P respectively. Prove that PC_0 and QB_0 meet on ω .

P21 The areas of rectangles P and Q are equal, but the diagonal of P is greater. Rectangle Q can be covered by two copies of P. Prove that P can be covered by two copies of Q.

P22 Let M_A, M_B, M_C be the midpoints of the sides BC, CA, AB respectively of a non-isosceles triangle ABC. Points H_A, H_B, H_C lie on the corresponding sides, different from M_A, M_B, M_C such that $M_AH_B = M_AH_C$, $M_BH_A = M_BH_C$, and $M_CH_A = M_CH_B$. Prove that H_A, H_B, H_C are the feet of the corresponding altitudes.

P23 A sphere touches all edges of a tetrahedron. Let a, b, c and d be the segments of the tangents to the sphere from the vertices of the tetrahedron. Is it true that that some of these segments necessarily form a triangle?

(It is not obligatory to use all segments. The side of the triangle can be formed by two segments)

– Grade 11

P24 A sphere is inscribed into a prism $ABCA'B'C'$ and touches its lateral faces $BCC'B', CAA'C', ABB'A'$ at points A_o, B_o, C_o respectively. It is known that $\angle A_oBB' = \angle B_oCC' = \angle C_oAA'$.
a) Find all possible values of these angles.
b) Prove that segments AA_o, BB_o, CC_o concur.
c) Prove that the projections of the incenter to $A'B', B'C', C'A'$ are the vertices of a regular triangle.