Spain Mathematical Olympiad 2022

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– Day 1

1 The six-pointed star in the figure is regular: all interior angles of the small triangles are equal. Each of the thirteen marked points is assigned a color, green or red. Prove that there are always three points of the same color, which are the vertices of an equilateral triangle.

2 Let a, b, c, d be four positive real numbers. If they satisfy

$$a+b+\frac{1}{ab}=c+d+\frac{1}{cd} \quad \text{and} \quad \frac{1}{a}+\frac{1}{b}+ab=\frac{1}{c}+\frac{1}{d}+cd$$

then prove that at least two of the values a, b, c, d are equal.

3 Let ABC be a triangle, with $AB < AC$, and let Γ be its circumcircle. Let D, E and F be the tangency points of the incircle with BC, CA and AB respectively. Let R be the point in EF such that DR is an altitude in the triangle DEF, and let S be the intersection of the external bisector of $\angle BAC$ with Γ. Prove that AR and SD intersect on Γ.

– Day 2

4 Let P be a point in the plane. Prove that it is possible to draw three rays with origin in P with the following property: for every circle with radius r containing P in its interior, if P_1, P_2 and P_3 are the intersection points of the three rays with the circle, then

$$|PP_1|+|PP_2|+|PP_3| \leq 3r.$$

5 Given is a simple graph G with 2022 vertices, such that for any subset S of vertices (including the set of all vertices), there is a vertex v with $deg_S(v) \leq 100$. Find $\chi(G)$ and the maximal number of edges G can have.

6 Find all triples (x, y, z) of positive integers, with $z > 1$, satisfying simultaneously that

$$x \text{ divides } y+1, \quad y \text{ divides } z-1, \quad z \text{ divides } x^2+1.$$