Balkan MO 2022

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by augustin_p, alchemyst_, TechnoLenzer

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1 Let ABC be an acute triangle such that $CA \neq CB$ with circumcircle ω and circumcentre O. Let t_A and t_B be the tangents to ω at A and B respectively, which meet at X. Let Y be the foot of the perpendicular from O onto the line segment CX. The line through C parallel to line AB meets t_A at Z. Prove that the line YZ passes through the midpoint of the line segment AC.

Proposed by Dominic Yeo, United Kingdom

2 Let a, b and n be positive integers with $a > b$ such that all of the following hold:

i. a^{2021} divides n,
ii. b^{2021} divides n,
iii. 2022 divides $a - b$.

Prove that there is a subset T of the set of positive divisors of the number n such that the sum of the elements of T is divisible by 2022 but not divisible by 2022^2.

Proposed by Silouanos Brazitikos, Greece

3 Find all functions $f : (0, \infty) \to (0, \infty)$ such that

$$f(y(f(x))^3 + x) = x^3 f(y) + f(x)$$

for all $x, y > 0$.

Proposed by Jason Prodromidis, Greece

4 Consider an $n \times n$ grid consisting of n^2 until cells, where $n \geq 3$ is a given odd positive integer. First, Dionysus colours each cell either red or blue. It is known that a frog can hop from one cell to another if and only if these cells have the same colour and share at least one vertex. Then, Xanthias views the colouring and next places k frogs on the cells so that each of the n^2 cells can be reached by a frog in a finite number (possible zero) of hops. Find the least value of k for which this is always possible regardless of the colouring chosen by Dionysus.

Proposed by Tommy Walker Mackay, United Kingdom