ITAMO 2022

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by Experia

1 Determine for which positive integers n there exists a positive integer A such that
• A is divisible by 2022,
• the decimal expression of A contains only digits 0 and 7,
• the decimal expression of A contains *exactly* n times the digit 7.

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2 Let ABC be an acute triangle with $AB < AC$. Let then
• D be the foot of the bisector of the angle in A,
• E be the point on segment BC (different from B) such that $AB = AE$,
• F be the point on segment BC (different from B) such that $BD = DF$,
• G be the point on segment AC such that $AB = AG$.
Prove that the circumcircle of triangle EFG is tangent to line AC.

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3 In a mathematical competition $n = 10\,000$ contestants participate.
During the final party, in sequence, the first one takes $1/n$ of the cake, the second one takes $2/n$ of the remaining cake, the third one takes $3/n$ of the cake that remains after the first and the second contestant, and so on until the last one, who takes all of the remaining cake.
Determine which competitor takes the largest piece of cake.

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4 Alberto chooses 2022 integers $a_1, a_2, \ldots, a_{2022}$ (not necessarily positive and not necessarily distinct) and places them on a 2022×2022 table such that in the (i, j) cell is the number a_k, with $k = \max\{i, j\}$, as shown in figure (in which, for a better readability, we have denoted a_{2022} with a_n).
Barbara does not know the numbers Alberto has chosen, but knows how they are displaced in the table. Given a positive integer k, with $1 \leq k \leq 2022$, Barbara wants to determine the value of a_k (and she is not interested in determining the values of the other a_i's with $i \neq k$). To do so, Barbara is allowed to ask Alberto one or more questions, in each of which she demands the value of the sum of the numbers contained in the cells of a "path", where with the term "path" we indicate a sorted list of cells with the following characteristics:
• the path starts from the top left cell and finishes with the bottom right cell,
• the cells of the path are all distinct,
• two consecutive cells of the path share a common side.
Determine, as k varies, the minimum number of questions Barbara needs to find a_k.

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5 Robot "Mag-o-matic" manipulates 101 glasses, displaced in a row whose positions are numbered from 1 to 101. In each glass you can find a ball or not. Mag-o-matic only accepts elementary instructions of the form $(a; b, c)$, which it interprets as
"consider the glass in position a: if it contains a ball, then switch the glasses in positions b and

c (together with their own content), otherwise move on to the following instruction"
(it means that a, b, c are integers between 1 and 101, with b and c different from each other but not necessarily different from a). A *programme* is a finite sequence of elementary instructions, assigned at the beginning, that Mag-o-matic does one by one.
A subset $S \subseteq \{0, 1, 2, \ldots, 101\}$ is said to be *identifiable* if there exists a programme which, starting from any initial configuration, produces a final configuration in which the glass in position 1 contains a ball if and only if the number of glasses containing a ball is an element of S.
(a) Prove that the subset S of the odd numbers is identifiable.
(b) Determine all subsets S that are identifiable.

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6 Let ABC be a non-equilateral triangle and let R be the radius of its circumcircle. The incircle of ABC has I as its centre and is tangent to side CA in point D and to side CB in point E.
Let A_1 be the point on line EI such that $A_1I = R$, with I being between A_1 and E. Let B_1 be the point on line DI such that $B_1I = R$, with I being between B_1 and D. Let P be the intersection of lines AA_1 and BB_1.
(a) Prove that P belongs to the circumcircle of ABC.
(b) Let us now also suppose that $AB = 1$ and P coincides with C. Determine the possible values of the perimeter of ABC.
