

**Balkan MO Shortlist 2021**

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- A1** Find all functions  $f : \mathbb{R}^+ \rightarrow \mathbb{R}$  and  $g : \mathbb{R}^+ \rightarrow \mathbb{R}$  such that

$$f(x^2 + y^2) = g(xy)$$

holds for all  $x, y \in \mathbb{R}^+$ .

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- A2** Find all functions  $f : \mathbb{R} \rightarrow \mathbb{R}$  such that

$$f(x^2 + y) \geq \left(\frac{1}{x} + 1\right)f(y)$$

holds for all  $x \in \mathbb{R} \setminus \{0\}$  and all  $y \in \mathbb{R}$ .

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- A3** Find all functions  $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ , such that  $f(x + f(x) + f(y)) = 2f(x) + y$  for all positive reals  $x, y$ .

*Proposed by Athanasios Kontogeorgis, Greece*

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- A4** Let  $f, g$  be functions from the positive integers to the integers. Vlad the impala is jumping around the integer grid. His initial position is  $x_0 = (0, 0)$ , and for every  $n \geq 1$ , his jump is

$$x_n - x_{n-1} = (\pm f(n), \pm g(n)) \text{ or } (\pm g(n), \pm f(n)),$$

with eight possibilities in total. Is it always possible that Vlad can choose his jumps to return to his initial location  $(0, 0)$  infinitely many times when

- (a)  $f, g$  are polynomials with integer coefficients?  
 (b)  $f, g$  are any pair of functions from the positive integers to the integers?
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- A5** Find all functions  $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  such that

$$f(xf(x + y)) = yf(x) + 1$$

holds for all  $x, y \in \mathbb{R}^+$ .

*Proposed by Nikola Velov, North Macedonia*

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- A6** Find all functions  $f : \mathbb{R} \rightarrow \mathbb{R}$  such that

$$f(xy) = f(x)f(y) + f(f(x + y))$$

holds for all  $x, y \in \mathbb{R}$ .

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- C1** Let  $\mathcal{A}_n$  be the set of  $n$ -tuples  $x = (x_1, \dots, x_n)$  with  $x_i \in \{0, 1, 2\}$ . A triple  $x, y, z$  of distinct elements of  $\mathcal{A}_n$  is called *good* if there is some  $i$  such that  $\{x_i, y_i, z_i\} = \{0, 1, 2\}$ . A subset  $A$  of  $\mathcal{A}_n$  is called *good* if every three distinct elements of  $A$  form a good triple. Prove that every good subset of  $\mathcal{A}_n$  has at most  $2\left(\frac{3}{2}\right)^n$  elements.

- C2** Let  $K$  and  $N > K$  be fixed positive integers. Let  $n$  be a positive integer and let  $a_1, a_2, \dots, a_n$  be distinct integers. Suppose that whenever  $m_1, m_2, \dots, m_n$  are integers, not all equal to 0, such that  $|m_i| \leq K$  for each  $i$ , then the sum

$$\sum_{i=1}^n m_i a_i$$

is not divisible by  $N$ . What is the largest possible value of  $n$ ?

*Proposed by Ilija Jovcevski, North Macedonia*

- C3** In an exotic country, the National Bank issues coins that can take any value in the interval  $[0, 1]$ . Find the smallest constant  $c > 0$  such that the following holds, no matter the situation in that country:

[i]Any citizen of the exotic country that has a finite number of coins, with a total value of no more than 1000, can split those coins into 100 boxes, such that the total value inside each box is at most  $c$ . [i]

- C4** A sequence of  $2n + 1$  non-negative integers  $a_1, a_2, \dots, a_{2n+1}$  is given. There's also a sequence of  $2n + 1$  consecutive cells enumerated from 1 to  $2n + 1$  from left to right, such that initially the number  $a_i$  is written on the  $i$ -th cell, for  $i = 1, 2, \dots, 2n + 1$ . Starting from this initial position, we repeat the following sequence of steps, as long as it's possible:

*Step 1:* Add up the numbers written on all the cells, denote the sum as  $s$ .

*Step 2:* If  $s$  is equal to 0 or if it is larger than the current number of cells, the process terminates. Otherwise, remove the  $s$ -th cell, and shift all cells that are to the right of it one position to the left. Then go to Step 1.

Example:  $(1, 0, 1, \underline{2}, 0) \rightarrow (1, \underline{0}, 1, 0) \rightarrow (1, \underline{1}, 0) \rightarrow (\underline{1}, 0) \rightarrow (0)$ .

A sequence  $a_1, a_2, \dots, a_{2n+1}$  of non-negative integers is called *balanced*, if at the end of this process there's exactly one cell left, and it's the cell that was initially enumerated by  $(n + 1)$ , i.e. the cell that was initially in the middle.

Find the total number of balanced sequences as a function of  $n$ .

*Proposed by Viktor Simjanoski, North Macedonia*

- C5** Problem 4. Angel has a warehouse, which initially contains 100 piles of 100 pieces of rubbish each. Each morning, Angel performs exactly one of the following moves:

- (a) He clears every piece of rubbish from a single pile.  
 (b) He clears one piece of rubbish from each pile.

However, every evening, a demon sneaks into the warehouse and performs exactly one of the following moves:

- (a) He adds one piece of rubbish to each non-empty pile.  
 (b) He creates a new pile with one piece of rubbish.

What is the first morning when Angel can guarantee to have cleared all the rubbish from the warehouse?

- C6** There is a population  $P$  of 10000 bacteria, some of which are friends (friendship is mutual), so that each bacterion has at least one friend and if we wish to assign to each bacterion a coloured membrane so that no two friends have the same colour, then there is a way to do it with 2021 colours, but not with 2020 or less.  
 Two friends  $A$  and  $B$  can decide to merge in which case they become a single bacterion whose friends are precisely the union of friends of  $A$  and  $B$ . (Merging is not allowed if  $A$  and  $B$  are not friends.) It turns out that no matter how we perform one merge or two consecutive merges, in the resulting population it would be possible to assign 2020 colours or less so that no two friends have the same colour. Is it true that in any such population  $P$  every bacterium has at least 2021 friends?

- G1** Let  $ABC$  be a triangle with  $AB < AC < BC$ . On the side  $BC$  we consider points  $D$  and  $E$  such that  $BA = BD$  and  $CE = CA$ . Let  $K$  be the circumcenter of triangle  $ADE$  and let  $F, G$  be the points of intersection of the lines  $AD, KC$  and  $AE, KB$  respectively. Let  $\omega_1$  be the circumcircle of triangle  $KDE$ ,  $\omega_2$  the circle with center  $F$  and radius  $FE$ , and  $\omega_3$  the circle with center  $G$  and radius  $GD$ .  
 Prove that  $\omega_1, \omega_2$ , and  $\omega_3$  pass through the same point and that this point of intersection lies on the line  $AK$ .

- G2** Let  $I$  and  $O$  be the incenter and the circumcenter of a triangle  $ABC$ , respectively, and let  $s_a$  be the exterior bisector of angle  $\angle BAC$ . The line through  $I$  perpendicular to  $IO$  meets the lines  $BC$  and  $s_a$  at points  $P$  and  $Q$ , respectively. Prove that  $IQ = 2IP$ .

- G3** Let  $ABC$  be a triangle with  $AB < AC$ . Let  $\omega$  be a circle passing through  $B, C$  and assume that  $A$  is inside  $\omega$ . Suppose  $X, Y$  lie on  $\omega$  such that  $\angle BXA = \angle AYC$ . Suppose also that  $X$  and  $C$  lie on opposite sides of the line  $AB$  and that  $Y$  and  $B$  lie on opposite sides of the line  $AC$ . Show that, as  $X, Y$  vary on  $\omega$ , the line  $XY$  passes through a fixed point.

*Proposed by Aaron Thomas, UK*

- G4** Let  $ABC$  be a right-angled triangle with  $\angle BAC = 90^\circ$ . Let the height from  $A$  cut its side  $BC$  at  $D$ . Let  $I, I_B, I_C$  be the incenters of triangles  $ABC, ABD, ACD$  respectively. Let also  $EB, EC$

be the excenters of  $ABC$  with respect to vertices  $B$  and  $C$  respectively. If  $K$  is the point of intersection of the circumcircles of  $E_CIB_I$  and  $E_BIC_I$ , show that  $KI$  passes through the midpoint  $M$  of side  $BC$ .

- G5** Let  $ABC$  be an acute triangle with  $AC > AB$  and circumcircle  $\Gamma$ . The tangent from  $A$  to  $\Gamma$  intersects  $BC$  at  $T$ . Let  $M$  be the midpoint of  $BC$  and let  $R$  be the reflection of  $A$  in  $B$ . Let  $S$  be a point so that  $SABT$  is a parallelogram and finally let  $P$  be a point on line  $SB$  such that  $MP$  is parallel to  $AB$ . Given that  $P$  lies on  $\Gamma$ , prove that the circumcircle of  $\triangle STR$  is tangent to line  $AC$ .

*Proposed by Sam Bealing, United Kingdom*

- G6** Let  $ABC$  be an acute triangle such that  $AB < AC$ . Let  $\omega$  be the circumcircle of  $ABC$  and assume that the tangent to  $\omega$  at  $A$  intersects the line  $BC$  at  $D$ . Let  $\Omega$  be the circle with center  $D$  and radius  $AD$ . Denote by  $E$  the second intersection point of  $\omega$  and  $\Omega$ . Let  $M$  be the midpoint of  $BC$ . If the line  $BE$  meets  $\Omega$  again at  $X$ , and the line  $CX$  meets  $\Omega$  for the second time at  $Y$ , show that  $A, Y$ , and  $M$  are collinear.

*Proposed by Nikola Velov, North Macedonia*

- G7** Let  $ABC$  be an acute scalene triangle. Its  $C$ -excircle tangent to the segment  $AB$  meets  $AB$  at point  $M$  and the extension of  $BC$  beyond  $B$  at point  $N$ . Analogously, its  $B$ -excircle tangent to the segment  $AC$  meets  $AC$  at point  $P$  and the extension of  $BC$  beyond  $C$  at point  $Q$ . Denote by  $A_1$  the intersection point of the lines  $MN$  and  $PQ$ , and let  $A_2$  be defined as the point, symmetric to  $A$  with respect to  $A_1$ . Define the points  $B_2$  and  $C_2$ , analogously. Prove that  $\triangle ABC$  is similar to  $\triangle A_2B_2C_2$ .

- G8** Let  $ABC$  be a scalene triangle and let  $I$  be its incenter. The projections of  $I$  on  $BC, CA$ , and  $AB$  are  $D, E$  and  $F$  respectively. Let  $K$  be the reflection of  $D$  over the line  $AI$ , and let  $L$  be the second point of intersection of the circumcircles of the triangles  $BFK$  and  $CEK$ . If  $\frac{1}{3}BC = AC - AB$ , prove that  $DE = 2KL$ .

- N1** Let  $n \geq 2$  be an integer and let

$$M = \left\{ \frac{a_1 + a_2 + \dots + a_k}{k} : 1 \leq k \leq n \text{ and } 1 \leq a_1 < \dots < a_k \leq n \right\}$$

be the set of the arithmetic means of the elements of all non-empty subsets of  $\{1, 2, \dots, n\}$ . Find

$$\min\{|a - b| : a, b \in M \text{ with } a \neq b\}.$$

- N2** Denote by  $l(n)$  the largest prime divisor of  $n$ . Let  $a_{n+1} = a_n + l(a_n)$  be a recursively defined sequence of integers with  $a_1 = 2$ . Determine all natural numbers  $m$  such that there exists some  $i \in \mathbb{N}$  with  $a_i = m^2$ .

*Proposed by Nikola Velov, North Macedonia*

**N3** Let  $n$  be a positive integer. Determine, in terms of  $n$ , the greatest integer which divides every number of the form  $p + 1$ , where  $p \equiv 2 \pmod{3}$  is a prime number which does not divide  $n$ .

**N4** Can every positive rational number  $q$  be written as

$$\frac{a^{2021} + b^{2023}}{c^{2022} + d^{2024}},$$

where  $a, b, c, d$  are all positive integers?

*Proposed by Dominic Yeo, UK*

**N5** A natural number  $n$  is given. Determine all  $(n - 1)$ -tuples of nonnegative integers  $a_1, a_2, \dots, a_{n-1}$  such that

$$\lfloor \frac{m}{2^n - 1} \rfloor + \lfloor \frac{2m + a_1}{2^n - 1} \rfloor + \lfloor \frac{2^2m + a_2}{2^n - 1} \rfloor + \lfloor \frac{2^3m + a_3}{2^n - 1} \rfloor + \dots + \lfloor \frac{2^{n-1}m + a_{n-1}}{2^n - 1} \rfloor = m$$

holds for all  $m \in \mathbb{Z}$ .

**N6** Let  $a, b$  and  $c$  be positive integers satisfying the equation  $(a, b) + [a, b] = 2021^c$ . If  $|a - b|$  is a prime number, prove that the number  $(a + b)^2 + 4$  is composite.

*Proposed by Serbia*

**N7** A *super-integer* triangle is defined to be a triangle whose lengths of all sides and at least one height are positive integers. We will deem certain positive integer numbers to be *good* with the condition that if the lengths of two sides of a super-integer triangle are two (not necessarily different) good numbers, then the length of the remaining side is also a good number. Let 5 be a good number. Prove that all integers larger than 2 are good numbers.