Brazilian Undergraduate Math Olympiad 2021
www.artofproblemsolving.com/community/c3043280
by Johann Peter Dirichlet

Problem 1 Consider the matrices like

$$M = \begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix}$$

such that $det(M) = 1$.

Show that

a) There are infinitely many matrices like above with $a, b, c \in \mathbb{Q}$

b) There are finitely many matrices like above with $a, b, c \in \mathbb{Z}$

Problem 2 Find all functions $f : \mathbb{R} \to \mathbb{R}$ from C^2 (id est, f is twice differentiable and f'' is continuous.) such that for every real number t we have $f(t)^2 = f(t\sqrt{2})$.

Problem 3 Find all positive integers k for which there is an irrational $\alpha > 1$ and a positive integer N such that $\lfloor \alpha^n \rfloor$ is of the form $m^2 - k$ com $m \in \mathbb{Z}$ for every integer $n > N$.

Problem 4 For every positive integeer $n > 1$, let $k(n)$ the largest positive integer k such that there exists a positive integer m such that $n = m^k$.

Find

$$lim_{n \to \infty} \frac{\sum_{j=2}^{j=n+1} k(j)}{n}$$

Problem 5 Find all triplets $(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^3$ such that there exists a matrix $A_{3\times 3}$ with all entries being non-negative reals whose eigenvalues are $\lambda_1, \lambda_2, \lambda_3$.

Problem 6 We recursively define a set of *goody pairs* of words on the alphabet $\{a, b\}$ as follows:

- (a, b) is a goody pair;
- $(\alpha, \beta) \neq (a, b)$ is a goody pair if and only if there is a goody pair (u, v) such that $(\alpha, \beta) = (uv, v)$ or $(\alpha, \beta) = (u, uv)$

Show that if (α, β) is a good pair then there exists a palindrome γ (possibly empty) such that $\alpha\beta = a\gamma b$
