Math olympiad for the French Speaking, 2022

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–	Juniors
1	find all the integer $n \geq 1$ such that $\lfloor \sqrt{n} \rfloor \mid n$
2	We consider an $n \times n$ table, with $n \geq 1$. Aya wishes to color k cells of this table so that that there is a unique way to place n tokens on colored squares without two tokens are not in the same row or column. What is the maximum value of k for which Aya's wish is achievable?
3	Let $\triangle ABC$ a triangle, and D the intersection of the angle bisector of $\angle BAC$ and the perpendicular bisector of AC. the line parallel to AC passing by the point B, intersect the line AD at X. the line parallel to CX passing by the point B, intersect AC at Y. $E = (AYB) \cap BX$. prove that C , D and E collinear.
4	find the smallest integer $n \geq 1$ such that the equation : $$a^2 + b^2 + c^2 - nd^2 = 0$$ has $(0, 0, 0, 0)$ as unique solution .
–	Seniors
1	find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that $f(m+n) + f(m)f(n) = n^2(f(m)+1) + m^2(f(n)+1) + mn(2-mn)$ holds for all $m, n \in \mathbb{Z}$
2	To connect to the OFM site, Alice must choose a password. The latter must be consisting of n characters among the following 27 characters: $$A, B, C, ..., Y, Z, \#$$ We say that a password m is *redundant* if we can color in red and blue a block of consecutive letters of m in such a way that the word formed from the red letters is identical to the word formed from blue letters. For example, the password $H\#ZBZJBJZ$ is redundant, because it contains the ZBZJBJ block, where the word ZBJ appears in both blue and red. At otherwise, the $ABCACB$ password is not redundant. Show that, for any integer $n \geq 1$, there exist at least 18^n passwords of length n, that is to say formed of n characters each, which are not redundant.

3 Let ABC be a triangle and Γ its circumcircle. Denote Δ the tangent at A to the circle Γ. Γ_1 is a circle tangent to the lines Δ, (AB) and (BC), and E its touchpoint with the line (AB). Let Γ_2 be a circle tangent to the lines Δ, (AC) and (BC), and F its touchpoint with the line (AC). We suppose that E and F belong respectively to the segments $[AB]$ and $[AC]$, and that the two circles Γ_1 and Γ_2 lie outside triangle ABC. Show that the lines (BC) and (EF) are parallel.

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4 find all positive integer $a \geq 2$ and $b \geq 2$ such that a is even and all the digits of $a^b + 1$ are equals.
