IV Iberoamerican Interuniversitary Mathematics Competition - Mexico

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by Ozc

Problem 1 For each positive integer n let A_n be the $n\times n$ matrix such that its a_{ij} entry is equal to $\binom{i+j-2}{j-1}$ for all $1\leq i,j\leq n$. Find the determinant of A_n.

Problem 2 A set $A\subset\mathbb{Z}$ is "padre" if whenever $x,y\in A$ with $x\leq y$ then also $2y-x\in A$. Prove that if A is "padre", $0,a,b\in A$ with $0<a<b$ and $d=g.c.d(a,b)$ then

$$a+b-3d,a+b-2d\in A.$$

Problem 3 Let $a,b,c,$ the lengths of the sides of a triangle. Prove that

$$\sqrt{\frac{(3a+b)(3b+a)}{(2a+c)(2b+c)}}+\sqrt{\frac{(3b+c)(3c+b)}{(2b+a)(2c+a)}}+\sqrt{\frac{(3c+a)(3a+c)}{(2c+b)(2a+b)}}\geq 4.$$

Problem 4 Let $f(x)=\frac{\sin(x)}{x}$ Find

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T\sqrt{1+f'(x)^2}dx.$$

Problem 5 Let $D=\{0,1,\dots,9\}$. A direction function for D is a function $f:D\times D\to\{0,1\}$. A real $r\in[0,1]$ is compatible with f if it can be written in the form

$$r=\sum_{j=1}^{\infty}\frac{d_j}{10^j}$$

with $d_j\in D$ and $f(d_j,d_{j+1})=1$ for every positive integer j.

Determine the least integer k such that for any direction fuction f, if there are k compatible reals with f then there are infinite reals compatible with f.

Problem 6 Let $n\geq 2$ and $p(x)=x^n+a_{n-1}x^{n-1}+\dots+a_1x+a_0$ a polynomial with real coefficients. Show that if there exists a positive integer k such that $(x-1)^{k+1}$ divides $p(x)$ then

$$\sum_{j=0}^{n-1}|a_j|>1+\frac{2k^2}{n}.$$
