V Iberoamerican Interuniversitary Mathematics Competition - Colombia

www.artofproblemsolving.com/community/c305023

by Ozc

Problem 1 Given two natural numbers m and n, denote by $\overline{m.n}$ the number obtained by writing m followed by n after the decimal dot.

a) Prove that there are infinitely many natural numbers k such that for any of them the equation $\overline{m.n} \times \overline{n.m} = k$ has no solution.

b) Prove that there are infinitely many natural numbers k such that for any of them the equation $\overline{m.n} \times \overline{n.m} = k$ has a solution.

Problem 2 Consider a polinomial $p \in \mathbb{R}[x]$ of degree n and with no real roots. Prove that

$$\int_{-\infty}^{\infty} \frac{(p'(x))^2}{(p(x))^2 + (p'(x))^2} dx$$

converges, and is less or equal than $n^{3/2}\pi$.

Problem 3 Given a set of boys and girls, we call a pair (A, B) amicable if A and B are friends. The friendship relation is symmetric. A set of people is affectionate if it satisfy the following conditions:

i) The set has the same number of boys and girls.

ii) For every four different people A, B, C, D if the pairs $(A, B), (B, C), (C, D)$ and (D, A) are all amicable, then at least one of the pairs (A, C) and (B, D) is also amicable.
iii) At least $\frac{1}{2013}$ of all boy-girl pairs are amicable.

Let m be a positive integer. Prove that there exists an integer $N(m)$ such that if a affectionate set has al least $N(m)$ people, then there exists m boys that are pairwise friends or m girls that are pairwise friends.

Problem 4 Let $a_1, b_1, c_1, a_2, b_2, c_2$ be positive real number and $F, G : (0, \infty) \to (0, \infty)$ be to differentiable and positive functions that satisfy the identities:

$$\frac{x}{F} = 1 + a_1x + b_1y + c_1G$$

$$\frac{y}{G} = 1 + a_2x + b_2y + c_2F.$$

Prove that if $0 < x_1 \le x_2$ and $0 < y_2 \le y_1$, then $F(x_1, x_2) \le F(x_2, y_2)$ and $G(x_1, y_1) \ge G(x_2, y_2)$.

Problem 5 Let A, B be $n \times n$ matrices with complex entries. Show that there exists a matrix T and an invertible matrix S such that

$$B = S(A+T)S^{-1} - T \iff \text{tr}(A) = \text{tr}(B)$$

Problem 6 Let (X, d) be a metric space with $d : X \times X \to \mathbb{R}_{\geq 0}$. Suppose that X is connected and compact. Prove that there exists an $\alpha \in \mathbb{R}_{\geq 0}$ with the following property: for any integer $n > 0$ and any $x_1, \dots, x_n \in X$, there exists $x \in X$ such that the average of the distances from $x_1, \dots, x_n$ to x is α i.e.

$$\frac{d(x, x_1) + d(x, x_2) + \cdots + d(x, x_n)}{n} = \alpha.$$
