VI Iberoamerican Interuniversitary Mathematics Competition - Costa Rica

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by Ozc

Problem 1 Let $g : [2013, 2014] \to \mathbb{R}$ a function that satisfy the following two conditions:

i) $g(2013) = g(2014) = 0,$
ii) for any $a, b \in [2013, 2014]$ it hold that $g\left(\frac{a+b}{2}\right) \leq g(a) + g(b).$
Prove that g has zeros in any open subinterval $(c, d) \subset [2013, 2014].$

Problem 2 Let n be an integer and p a prime greater than 2. Show that:

$$(p-1)^n n! | (p^n - 1)(p^n - p)(p^n - p^2) \cdots (p^n - p^{n-1}).$$

Problem 3 Given $n \geq 2$, let $\mathcal{A}$ be a family of subsets of the set $\{1, 2, \dots, n\}$ such that, for any $A_1, A_2, A_3, A_4 \in \mathcal{A}$, it holds that $|A_1 \cup A_2 \cup A_3 \cup A_4| \leq n - 2.$
Prove that $|\mathcal{A}| \leq 2^{n-2}.$

Problem 4 Let $\{a_i\}$ be a strictly increasing sequence of positive integers. Define the sequence $\{s_k\}$ as

$$s_k = \sum_{i=1}^{k} \frac{1}{[a_i, a_{i+1}]},$$

where $[a_i, a_{i+1}]$ is the least commun multiple of a_i and $a_{i+1}.$
Show that the sequence $\{s_k\}$ is convergent.

Problem 5 A analityc function $f : \mathbb{C} \to \mathbb{C}$ is call interesting if $f(z)$ is real along the parabola $Re(z) = (Im(z))^2.$

a) Find an example of a non constant interesting function.
b) Show that every interesting function f satisfy that $f'(-3/4) = 0.$

Problem 6 a) Let $\{x_n\}$ be a sequence with $x_n \in [0, 1]$ for any n. Prove that there exists $C > 0$ such that for every positive integer r, there exists $m \geq 1$ and $n > m + r$ that satisfy $(n - m)|x_n - x_m| \leq C.$
b) Prove that for every $C > 0$, there exists a sequence $\{x_n\}$ with $x_n \in [0, 1]$ for all n and an integer r such that, if $m \geq 1$ and $n > m + r$, then $(n - m)|x_n - x_m| > C.$
