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Test 1 November 2nd, 2020

1 Which is greater:

$$\frac{1^{-3}-2^{-3}}{1^{-2}-2^{-2}}-\frac{2^{-3}-3^{-3}}{2^{-2}-3^{-2}}+\frac{3^{-3}-4^{-3}}{3^{-2}-4^{-2}}-\dots+\frac{2019^{-3}-2020^{-3}}{2019^{-2}-2020^{-2}}$$

or

$$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\dots+\frac{1}{5781}?$$

2 Given 10 light switches, each can be in two states: on and off. For each pair of switches there is a light bulb which is on if and only if when both switches are on (45 bulbs in total). The bulbs and the switches are unmarked so it is unclear which switches correspond to which bulb. In the beginning all switches are off. How many flips are needed to find out regarding all bulbs which switches are connected to it? On each step you can flip precisely one switch

3 Let ABC be an acute triangle with orthocenter H. Prove that there is a line l which is parallel to BC and tangent to the incircles of ABH and ACH.

4 Let r be a positive integer and let a_r be the number of solutions to the equation $3^x-2^y=r$,such that $0\leq x,y\leq 5781$ are integers. What is the maximal value of a_r?

Test 2 November 4th, 2020

1 Let $ABCDEFGHIJ$ be a regular 10-gon. Let T be a point inside the 10-gon, such that the DTE is isosceles: $DT=ET$, and its angle at the apex is 72°. Prove that there exists a point S such that FTS and HIS are both isosceles, and for both of them the angle at the apex is 72°.

2 Suppose $x,y,z\in\mathbb{R}^+$. Prove that

$$\frac{x}{\sqrt{yz+4xy+4xz}}+\frac{y}{\sqrt{zx+4yz+4yx}}+\frac{z}{\sqrt{xy+4zx+4zy}}\geq 1$$

.

3 A game is played on a $n\times n$ chessboard. In the beginning Bars the cat occupies any cell according to his choice. The d sparrows land on certain cells according to their choice (several sparrows may land in the same cell). Bars and the sparrows play in turns. In each turn of Bars,

he moves to a cell adjacent by a side or a vertex (like a king in chess). In each turn of the sparrows, precisely one of the sparrows flies from its current cell to any other cell of his choice. The goal of Bars is to get to a cell containing a sparrow. Can Bars achieve his goal
a) if $d = \lfloor \frac{3 \cdot n^2}{25} \rfloor$, assuming n is large enough?
b) if $d = \lfloor \frac{3 \cdot n^2}{19} \rfloor$, assuming n is large enough?
c) if $d = \lfloor \frac{3 \cdot n^2}{14} \rfloor$, assuming n is large enough?

Test 4 February 17, 2021

1 Ayala and Barvaz play a game: Ayala initially gives Barvaz two 100×100 tables of positive integers, such that the product of numbers in each table is the same. In one move, Barvaz may choose a row or column in one of the tables, and change the numbers in it (to some positive integers), as long as the total product remains the same. Barvaz wins if after N such moves, he manages to make the two tables equal to each other, and otherwise Ayala wins.
a. For which values of N does Barvaz have a winning strategy?
b. For which values of N does Barvaz have a winning strategy, if all numbers in Ayalah's tables must be powers of 2?

2 Find all unbounded functions $f : \mathbb{Z} \to \mathbb{Z}$, such that $f(f(x) - y) | x - f(y)$ holds for any integers x, y.

3 Consider a triangle ABC and two congruent triangles $A_1B_1C_1$ and $A_2B_2C_2$ which are respectively similar to ABC and inscribed in it: A_i, B_i, C_i are located on the sides of ABC in such a way that the points A_i are on the side opposite to A, the points B_i are on the side opposite to B, and the points C_i are on the side opposite to C (and the angle at A are equal to angles at A_i etc.).
The circumcircles of $A_1B_1C_1$ and $A_2B_2C_2$ intersect at points P and Q. Prove that the line PQ passes through the orthocenter of ABC.

Test 6 May 11th, 2021

1 A pair of positive integers (a, b) is called an **average couple** if there exist positive integers k and $c_1, \dots, c_k$ for which

$$\frac{c_1 + c_2 + \cdots + c_k}{k} = a \qquad \text{and} \qquad \frac{s(c_1) + s(c_2) + \cdots + s(c_k)}{k} = b$$

where $s(n)$ denotes the sum of digits of n in decimal representation.
Find the number of average couples (a, b) for which $a, b < 10^{10}$.

2 Let $n > 1$ be an integer. Hippo chooses a list of n points in the plane $P_1, \dots, P_n$; some of these points may coincide, but not all of them can be identical. After this, Wombat picks a point from the list X and measures the distances from it to the other $n - 1$ points in the list. The average

of the resulting $n-1$ numbers will be denoted $m(X)$.
Find all values of n for which Hippo can prepare the list in such a way, that for any point X Wombat may pick, he can point to a point Y from the list such that $XY = m(X)$.

3 In an inscribed quadrilateral $ABCD$, we have $BC = CD$ but $AB \neq AD$. Points I and J are the incenters of triangles ABC and ACD respectively. Point K was chosen on segment AC so that $IK = JK$. Points M and N are the incenters of triangles AIK and AJK. Prove that the perpendicular to CD at D and the perpendicular to KI at I intersect on the circumcircle of MAN.

Test 8 June 9th, 2021

1 An ordered quadruple of numbers is called *ten-esque* if it is composed of 4 nonnegative integers whose sum is equal to 10. Ana chooses a ten-esque quadruple (a_1, a_2, a_3, a_4) and Banana tries to guess it. At each stage Banana offers a ten-esque quadtruple (x_1, x_2, x_3, x_4) and Ana tells her the value of

$$|a_1 - x_1| + |a_2 - x_2| + |a_3 - x_3| + |a_4 - x_4|$$

How many guesses are needed for Banana to figure out the quadruple Ana chose?

2 Find all functions $f : \mathbb{R} \to \mathbb{R}$ so that for any reals x, y the following holds:

$$f(x \cdot f(x+y)) + f(f(y) \cdot f(x+y)) = (x+y)^2$$

3 What is the smallest value of k for which the inequality

$$ad - bc + yz - xt + (a+c)(y+t) - (b+d)(x+z) \leq$$
$$\leq k\left(\sqrt{a^2+b^2} + \sqrt{c^2+d^2} + \sqrt{x^2+y^2} + \sqrt{z^2+t^2}\right)^2$$

holds for any 8 real numbers a, b, c, d, x, y, z, t?

Edit: Fixed a mistake! Thanks @below.