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1 Prove that for all positive real numbers a, b the following inequality holds:

$$\sqrt{\frac{a^2+b^2}{2}}+\frac{2ab}{a+b}\geq\frac{a+b}{2}+\sqrt{ab}$$

When does equality hold?

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2 Let I be the incenter, A_1 and B_1 midpoints of sides BC and AC of a triangle ΔABC. Denote by M and N the midpoints of the arcs AC and BC of circumcircle of ΔABC which do contain the other vertex of the triangle. If points M, I and N are collinear prove that:

$$\angle AIB_1=\angle BIA_1=90^\circ$$

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3 Find all natural numbers n for which the following 5 conditions hold: (1) n is not divisible by any perfect square bigger than 1. (2) n has exactly one prime divisor of the form $4k+3$, $k\in\mathbb{N}_0$. (3) Denote by $S(n)$ the sum of digits of n and $d(n)$ as the number of positive divisors of n. Then we have that $S(n)+2=d(n)$. (4) $n+3$ is a perfect square. (5) n does not have a prime divisor which has 4 or more digits.

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4 Initially in every cell of a 5×5 board is the number 0. In one move you may take any cell of this board and add 1 to it and all of its adjacent cells (two cells are adjacent if they share an edge). After a finite number of moves, number n is written in all cells. Find all possible values of n.
