OMMC Year 2 2021-2022 problems

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by squareman, pog, StopSine

–	Main Competition
1	The integers from 1 through 9 inclusive, are placed in the squares of a 3×3 grid. Each square contains a different integer. The product of the integers in the first and second rows are 60 and 96 respectively. Find the sum of the integers in the third row. *Proposed by bissue*
2	In a room, each person is an painter and/or a musician. 2 percent of the painters are musicians, and 5 percent of the musicians are painters. Only one person is both an painter and a musician. How many people are in the room? *Proposed by Evan Chang*
3	Evan has 10 cards numbered 1 through 10. He chooses some of the cards and takes the product of the numbers on them. When the product is divided by 3, the remainder is 1. Find the maximum number of cards he could have chose. *Proposed by Evan Chang*
4	How many sequences of real numbers $a_1, a_2, \dots a_9$ satisfy $$\lvert a_1 - 1\rvert = \lvert a_2 - a_1\rvert = \cdots = \lvert a_9 - a_8\rvert = \lvert 1 - a_9\rvert = 1?$$ *Proposed by Evan Chang*
5	12 distinct points are equally spaced around a circle. How many ways can Bryan choose 3 points (not in any order) out of these 12 points such that they form an acute triangle (Rotations of a set of points are considered distinct). *Proposed by Bryan Guo*
6	Calvin makes a number. He starts with 1, and on each move, he multiplies his current number by 3, then adds 5. After 10 moves, find the sum of the digits (in base 10) when Calvin's resulting number is expressed in base 9. *Proposed by Calvin Wang*

7 How many ordered triples of integers (x, y, z) satisfy

$$36x^2 + 100y^2 + 225z^2 = 12600?$$

Proposed by Bill Fei and Mahith Gottipati

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8 Isaac repeatedly flips a fair coin. Whenever a particular face appears for the $2n+1$th time, for any nonnegative integer n, he earns a point. The expected number of flips it takes for Isaac to get 10 points is $\frac{a}{b}$ for coprime positive integers a and b. Find $a+b$.

Proposed by Isaac Chen

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9 12 people stand in a row. Each person is given a red shirt or a blue shirt. Every minute, exactly one pair of two people with the same color currently standing next to each other in the row leave. After 6 minutes, everyone has left. How many ways could the shirts have been assigned initially?

Proposed by Evan Chang

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10 A real number x satisfies $2 + \log_{25} x + \log_8 5 = 0$. Find

$$\log_2 x - (\log_8 5)^3 - (\log_{25} x)^3.$$

Proposed by Evan Chang

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11 Let ABC be a triangle such that $AB = 7$, $BC = 8$, and $CA = 9$. There exists a unique point X such that $XB = XC$ and XA is tangent to the circumcircle of ABC. If $XA = \frac{a}{b}$, where a and b are coprime positive integers, find $a+b$.

Proposed by Alexander Wang

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12 Katelyn is building an integer (in base 10). She begins with 9. Each step, she appends a randomly chosen digit from 0 to 9 inclusive to the right end of her current integer. She stops immediately when the current integer is 0 or 1 (mod 11). The probability that the final integer ends up being 0 (mod 11) is $\frac{a}{b}$ for coprime positive integers a, b. Find $a+b$.

Proposed by Evan Chang

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13 $ABCD$ is a rhombus where $\angle BAD = 60^\circ$. Point E lies on minor arc $\widehat{AD}$ of the circumcircle of ABD, and F is the intersection of AC and the circle circumcircle of EDC. If $AF = 4$ and the circumcircle of EDC has radius 14, find the squared area of $ABCD$.

Proposed by Vivian Loh

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14 The corners of a 2-dimensional room in the shape of an isosceles right triangle are labeled A, B, C where $AB = BC$. Walls BC and CA are mirrors. A laser is shot from A, hits off of each of the mirrors once and lands at a point X on AB. Let Y be the point where the laser hits off AC. If $\frac{AB}{AX} = 64$, $\frac{CA}{AY} = \frac{p}{q}$ for coprime positive integers p, q. Find $p + q$.

Proposed by Sid Doppalapudi

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15 Let $1 = x_1 < x_2 < \cdots < x_k = n$ denote the sequence of all divisors $x_1, x_2 \dots x_k$ of n in increasing order. Find the smallest possible value of n such that

$$n = x_1^2 + x_2^2 + x_3^2 + x_4^2.$$

Proposed by Justin Lee

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16 In $\triangle ABC$ with $AB = 10$, $BC = 12$, and $AC = 14$, let E and F be the midpoints of AB and AC. If a circle passing through B and C is tangent to the circumcircle of AEF at point $X \neq A$, find AX.

Proposed by Vivian Loh

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17 Find the number of positive integer divisors of

$$\sum_{k=0}^{50} (-3)^k \binom{100}{2k}.$$

Proposed by Serena Xu

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18 Define mutually externally tangent circles ω_1, ω_2, and ω_3. Let ω_1 and ω_2 be tangent at P. The common external tangents of ω_1 and ω_2 meet at Q. Let O be the center of ω_3. If $QP = 420$ and $QO = 427$, find the radius of ω_3.

Proposed by Tanishq Pauskar and Mahith Gottipati

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19 N people have a series of calls. Each call is between two people, and is started by exactly one of them. Each person starts at most 10 calls. Two people can call at most once. In any group of 3 people, there are at least two people who have a call. Find the maximum possible value of N.

Proposed by Serena Xu

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20 Let

$$\mathcal{S} = \sum_{i=1}^{\infty} \left(\prod_{j=1}^{i} \frac{3j-2}{12j} \right).$$

Then $(\mathcal{S} + 1)^3 = \frac{m}{n}$ with m and n coprime positive integers. Find $10m + n$.

Proposed by Justin Lee and Evan Chang

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21 For some real number a, define two parabolas on the coordinate plane with equations $x = y^2 + a$ and $y = x^2 + a$. Suppose there are 3 lines, each tangent to both parabolas, that form an equilateral triangle with positive area s. If $s^2 = \frac{p}{q}$ for coprime positive integers p, q, find $p + q$.

Proposed by Justin Lee

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22 A positive integer N is *apt* if for each integer $0 < k < 1009$, there exists exactly one divisor of N with a remainder of k when divided by 1009. For a prime p, suppose there exists an *apt* positive integer N where $\frac{N}{p}$ is an integer but $\frac{N}{p^2}$ is not. Find the number of possible remainders when p is divided by 1009.

Proposed by Evan Chang

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23 A 39-tuple of real numbers $(x_1, x_2, \dots x_{39})$ satisfies

$$2\sum_{i=1}^{39} \sin(x_i) = \sum_{i=1}^{39} \cos(x_i) = -34.$$

The ratio between the maximum of $\cos(x_1)$ and the maximum of $\sin(x_1)$ over all tuples $(x_1, x_2, \dots x_{39})$ satisfying the condition is $\frac{a}{b}$ for coprime positive integers a, b (these maxima aren't necessarily achieved using the same tuple of real numbers). Find $a + b$.

Proposed by Evan Chang

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24 In $\triangle ABC$, angle B is obtuse, $AB = 42$ and $BC = 69$. Let M and N be the midpoints of AB and BC, respectively. The angle bisectors of $\angle CAB$ and $\angle ABC$ meet BC and CA at D and E respectively. Let X and Y be the midpoints of AD and AN respectively. Let CY and BX meet AB and CA at P and Q. If EM and PQ meet on BC, find CA.

Proposed by Sid Doppalapudi

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25 Let $K > 0$ be an integer. An integer $k \in [0, K]$ is randomly chosen. A sequence of integers is defined starting on k and ending on 0, where each nonzero term t is followed by t minus the largest Lucas number not exceeding t.

The probability that 4, 5, or 6 is in this sequence approaches $\frac{a-b\sqrt{c}}{d}$ for arbitrarily large K, where a, b, c, d, are positive integers, $\gcd(a, b, d) = 1$, and c is squarefree. Find $a + b + c + d$.

[i](Lucas numbers are defined as the members of the infinite integer sequence 2, 1, 3, 4, 7, $\dots$ where each term is the sum of the two before it.)[/i]

Proposed by Evan Chang

– Tiebreaker

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1 Find the sum of all positive integers n where the mean and median of $\{20, 42, 69, n\}$ are both integers.

Proposed by bissue

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2 Alex writes down some distinct integers on a blackboard. For each pair of integers, he writes the positive difference of those on a piece of paper. Find the sum of all $n \leq 2022$ such that it is possible for the numbers on the paper to contain only the positive integers between 1 and n, inclusive exactly once.

Proposed by Alexander Wang

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3 Parabolas P_1, P_2 share a focus at $(20, 22)$ and their directrices are the x and y axes respectively. They intersect at two points X, Y. Find XY^2.

Proposed by Evan Chang

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4 If x, y, z satisfy $x+y+z=12$, $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=2$ and $x^3+y^3+z^3=-480$, find

$$x^2y+xy^2+x^2z+xz^2+y^2z+yz^2.$$

Proposed by Mahith Gottipati

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5 A frog starts a journey at $(6, 9)$. A skip is the act of traveling a positive integer number of units straight south or a positive integer number of units straight west. A jump is the act of traveling one unit straight west. A hop consists of any skip followed by a jump. How many different sequences of hops can the frog take so that the frog's final destination is $(0, 0)$?

Proposed by Jack Ma
