Problems from 2022 Thailand Mathematical Olympiad

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by Quidditch

– Day 1

1 Let BC be a chord of a circle Γ and A be a point inside Γ such that $\angle BAC$ is acute. Outside $\triangle ABC$, construct two isosceles triangles $\triangle ACP$ and $\triangle ABR$ such that $\angle ACP$ and $\angle ABR$ are right angles. Let lines BA and CA meet Γ again at points E and F, respectively. Let lines EP and FR meet Γ again at points X and Y, respectively. Prove that $BX = CY$.

2 Define a function $f : \mathbb{N} \times \mathbb{N} \to \{-1, 1\}$ such that

$$f(m,n) = \begin{cases} 1 & \text{if } m, n \text{ have the same parity, and} \\ -1 & \text{if } m, n \text{ have different parity} \end{cases}$$

for every positive integers m, n. Determine the minimum possible value of

$$\sum_{1 \le i < j \le 2565} ij f(x_i, x_j)$$

across all permutations $x_1, x_2, x_3, \dots, x_{2565}$ of $1, 2, \dots, 2565$.

3 Let Ω be a circle in a plane. 2022 pink points and 2565 blue points are placed inside Ω such that no point has two colors and no two points are collinear with the center of Ω. Prove that there exists a sector of Ω such that the angle at the center is acute and the number of blue points inside the sector is greater than the number of pink points by exactly 100. (Note: such sector may contain no pink points.)

4 Find all positive integers n such that there exists a monic polynomial $P(x)$ of degree n with integers coefficients satisfying

$$P(a)P(b) \neq P(c)$$

for all integers a, b, c.

5 Determine all functions $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ that satisfies the equation

$$f\left(\frac{x+y+z}{3}, \frac{a+b+c}{3}\right) = f(x,a)f(y,b)f(z,c)$$

for any real numbers x, y, z, a, b, c such that $az + bx + cy \neq ay + bz + cx$.

– Day 2

6 In an examination, there are 3600 students sitting in a 60×60 grid, where everyone is facing toward the top of the grid. After the exam, it is discovered that there are 901 students who got infected by COVID-19. Each infected student has a spreading region, which consists of students to the left, to the right, or in the front of them. Student in spreading region of at least two students are considered a close contact. Given that no infected student sat in the spreading region of other infected student, prove that there is at least one close contact.

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7 Let $d \geq 2$ be a positive integer. Define the sequence $a_1, a_2, \dots$ by

$$a_1 = 1 \text{ and } a_{n+1} = a_n^d + 1 \text{ for all } n \geq 1.$$

Determine all pairs of positive integers (p, q) such that a_p divides a_q.

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8 Determine all possible values of a_1 for which there exists a sequence $a_1, a_2, \dots$ of rational numbers satisfying

$$a_{n+1}^2 - a_{n+1} = a_n$$

for all positive integers n.

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9 Let $PQRS$ be a quadrilateral that has an incircle and $PQ \neq QR$. Its incircle touches sides $PQ, QR, RS,$ and SP at $A, B, C,$ and D, respectively. Line RP intersects lines BA and BC at T and M, respectively. Place point N on line TB such that NM bisects $\angle TMB$. Lines CN and TM intersect at K, and lines BK and CD intersect at H. Prove that $\angle NMH = 90^\circ$.

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10 For each positive integers u and n, say that u is a *friend* of n if and only if there exists a positive integer N that is a multiple of n and the sum of digits of N (in base 10) is equal to u. Determine all positive integers n that only finitely many positive integers are not a friend of n.
