**German National Olympiad 2022, Final Round**

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**–** Day 1

**1** Determine all real numbers $a$ for which the system of equations

$$3x^2 + 2y^2 + 2z^2 = a$$
$$4x^2 + 4y^2 + 5z^2 = 1 - a$$

has at least one solution $(x, y, z)$ in the real numbers.

**2** As everyone knows, the people of *Plane Land* love Planimetrics. Therefore, they imagine their country as completely planar, every city in the country as a geometric point and every road as the line segment connecting two points.
Additionally to the existing cities, it is possible to build *roundabouts*, i.e. points in the road network from where at least two roads emanate. All road crossings or junctions are build as roundabouts. Via this route network, every two cities should be connected by a sequence of roads and possibly roundabouts. In Plane Land, the length of a road is taken as the geometric length of the corresponding line segment.

The ingenious road engineer Armin Asphalt presents a new road map, of which it is known that there is no road network with a smaller total length of all roads. Moreover, there is no road map with the same total length of all roads and fewer roundabouts.
Prove that in the road map of Armin Asphalt, at most three roads emanate from each city, and exactly three from each roundabout.

**3** Let $M$ and $N$ be the midpoints of segments $BC$ and $AC$ of a triangle $ABC$, respectively. Let $Q$ be a point on the line through $N$ parallel to $BC$ such that $Q$ and $C$ are on opposite sides of $AB$ and $|QN| \cdot |BC| = |AB| \cdot |AC|$.

Suppose that the circumcircle of triangle $AQN$ intersects the segment $MN$ a second time in a point $T \neq N$.
Prove that there is a circle through points $T$ and $N$ touching both the side $BC$ and the incircle of triangle $ABC$.

**–** Day 2

**4** Determine all $6$-tuples $(x, y, z, u, v, w)$ of integers satisfying the equation

$$x^3 + 7y^3 + 49z^3 = 2u^3 + 14v^3 + 98w^3.$$

**5** Let $ABC$ be an equilateral triangle with circumcircle $k$. A circle $q$ touches $k$ from outside in a point $D$, where the point $D$ on $k$ is chosen so that $D$ and $C$ lie on different sides of the line $AB$. We now draw tangent lines from $A, B$ and $C$ to the circle $q$ and denote the lengths of the respective tangent line segments by $a, b$ and $c$.

Prove that $a + b = c$.

**6** Consider functions $f$ satisfying the following four conditions:
(1) $f$ is real-valued and defined for all real numbers.
(2) For any two real numbers $x$ and $y$ we have $f(xy) = f(x)f(y)$.
(3) For any two real numbers $x$ and $y$ we have $f(x+y) \leq 2(f(x)+f(y))$.
(4) We have $f(2) = 4$.

Prove that:
a) There is a function $f$ with $f(3) = 9$ satisfying the four conditions.
b) For any function $f$ satisfying the four conditions, we have $f(3) \leq 9$.