China Girls Math Olympiad 2016

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Day 1

1 Let $n\geq 3$ be an integer. Put n^2 cards, each labelled $1,2,\ldots ,n^2$ respectively, in any order into n empty boxes such that there are exactly n cards in each box. One can perform the following operation: one first selects 2 boxes, takes out any 2 cards from each of the selected boxes, and then return the cards to the other selected box. Prove that, for any initial order of the n^2 cards in the boxes, one can perform the operation finitely many times such that the labelled numbers in each box are consecutive integers.

2 In $\triangle ABC, BC=a,CA=b,AB=c$, and Γ is its circumcircle. (1) Determine a necessary and sufficient condition on a,b and c if there exists a unique point $P(P\neq B,P\neq C)$ on the arc BC of Γ not passing through point A such that $PA=PB+PC$. (2) Let P be the unique point stated in (1). If AP bisects BC, prove that $\angle BAC<60^{\circ}$.

3 Let m and n are relatively prime integers and $m>1,n>1$. Show that:There are positive integers a,b,c such that $m^a=1+n^bc$, and n and c are relatively prime.

4 Let n is a positive integers ,$a_1,a_2,\cdots,a_n\in\{0,1,\cdots,n\}$. For the integer j $(1\le j\le n)$,define b_j is the number of elements in the set $\{i|i\in\{1,\cdots,n\},a_i\ge j\}$ .For example When $n=3$,if $a_1=1,a_2=2,a_3=1$,then $b_1=3,b_2=1,b_3=0$. (1) Prove that

$$\sum_{i=1}^{n}(i+a_i)^2\ge \sum_{i=1}^{n}(i+b_i)^2.$$

(2) Prove that

$$\sum_{i=1}^{n}(i+a_i)^k\ge \sum_{i=1}^{n}(i+b_i)^k,$$

for the integer $k\ge 3$.

Day 2

5 Define a sequence $\{a_n\}$ by

$$S_1=1,\ S_{n+1}=\frac{(2+S_n)^2}{4+S_n}(n=1,\ 2,\ 3,\ \cdots).$$

Where S_n the sum of first n terms of sequence $\{a_n\}$.
For any positive integer n ,prove that

$$a_n \geq \frac{4}{\sqrt{9n+7}}.$$

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6 Find the greatest positive integer m, such that one of the 4 letters C, G, M, O can be placed in each cell of a table with m rows and 8 columns, and has the following property: For any two distinct rows in the table, there exists at most one column, such that the entries of these two rows in such a column are the same letter.

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7 In acute triangle $ABC, AB < AC$, I is its incenter, D is the foot of perpendicular from I to BC, altitude AH meets BI, CI at P, Q respectively. Let O be the circumcenter of $\triangle IPQ$, extend AO to meet BC at L. Circumcircle of $\triangle AIL$ meets BC again at N. Prove that $\frac{BD}{CD} = \frac{BN}{CN}$.

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8 Let $\mathbb{Q}$ be the set of rational numbers, $\mathbb{Z}$ be the set of integers. On the coordinate plane, given positive integer m, define

$$A_m = \left\{(x, y) \mid x, y \in \mathbb{Q}, xy \neq 0, \frac{xy}{m} \in \mathbb{Z}\right\}.$$

For segment MN, define $f_m(MN)$ as the number of points on segment MN belonging to set A_m.

Find the smallest real number λ, such that for any line l on the coordinate plane, there exists a constant $\beta(l)$ related to l, satisfying: for any two points M, N on l,

$$f_{2016}(MN) \leq \lambda f_{2015}(MN) + \beta(l)$$
