JBMO Shortlist 2021

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by Lukaluce

A1 Let n ($n \geq 1$) be an integer. Consider the equation

$2 \cdot \lfloor \frac{1}{2x} \rfloor - n + 1 = (n+1)(1-nx)$,

where x is the unknown real variable.

(a) Solve the equation for $n = 8$.
(b) Prove that there exists an integer n for which the equation has at least 2021 solutions.
(For any real number y by $\lfloor y \rfloor$ we denote the largest integer m such that $m \leq y$.)

A2 Let $n > 3$ be a positive integer. Find all integers k such that $1 \leq k \leq n$ and for which the following property holds:
If $x_1, ..., x_n$ are n real numbers such that $x_i + x_{i+1} + ... + x_{i+k-1} = 0$ for all integers $i > 1$ (indexes are taken modulo n), then $x_1 = ... = x_n = 0$.

Proposed by *Vincent Jugé and Théo Lenoir, France*

A3 Let n be a positive integer. A finite set of integers is called n-divided if there are exactly n ways to partition this set into two subsets with equal sums. For example, the set $\{1, 3, 4, 5, 6, 7\}$ is 2-divided because the only ways to partition it into two subsets with equal sums is by dividing it into $\{1, 3, 4, 5\}$ and $\{6, 7\}$, or $\{1, 5, 7\}$
and $\{3, 4, 6\}$. Find all the integers $n > 0$ for which there exists a n-divided set.

Proposed by *Martin Rakovsky, France*

G1 Let ABC be an acute scalene triangle with circumcenter O. Let D be the foot of the altitude from A to the side BC. The lines BC and AO intersect at E. Let s be the line through E perpendicular to AO. The line s intersects AB and AC at K and L, respectively. Denote by ω the circumcircle of triangle AKL. Line AD intersects ω again at X.
Prove that ω and the circumcircles of triangles ABC and DEX have a common point.

G2 Let P be an interior point of the isosceles triangle ABC with $\hat{A} = 90^\circ$. If

$$\widehat{PAB} + \widehat{PBC} + \widehat{PCA} = 90^\circ,$$

prove that $AP \perp BC$.

Proposed by *Mehmet Akif Yıldız, Turkey*

G3 Let ABC be an acute triangle with circumcircle ω and circumcenter O. The perpendicular from A to BC intersects BC and ω at D and E, respectively. Let F be a point on the segment AE, such that $2 \cdot FD = AE$. Let l be the perpendicular to OF through F. Prove that l, the tangent to ω at E, and the line BC are concurrent.

Proposed by *Stefan Lozanovski, Macedonia*

G4 Let $ABCD$ be a convex quadrilateral with $\angle B = \angle D = 90^\circ$. Let E be the point of intersection of BC with AD and let M be the midpoint of AE. On the extension of CD, beyond the point D, we pick a point Z such that $MZ = \frac{AE}{2}$. Let U and V be the projections of A and E respectively on BZ. The circumcircle of the triangle DUV meets again AE at the point L. If I is the point of intersection of BZ with AE, prove that the lines BL and CI intersect on the line AZ.

G5 Let ABC be an acute scalene triangle with circumcircle ω. Let P and Q be interior points of the sides AB and AC, respectively, such that PQ is parallel to BC. Let L be a point on ω such that AL is parallel to BC. The segments BQ and CP intersect at S. The line LS intersects ω at K. Prove that $\angle BKP = \angle CKQ$.

Proposed by *Ervin Macić, Bosnia and Herzegovina*

N1 Find all positive integers a, b, c such that $ab + 1$, $bc + 1$, and $ca + 1$ are all equal to factorials of some positive integers.

Proposed by *Nikola Velov, Macedonia*

N2 The real numbers x, y and z are such that $x^2 + y^2 + z^2 = 1$.
a) Determine the smallest and the largest possible values of $xy + yz - xz$.
b) Prove that there does not exist a triple (x, y, z) of rational numbers, which attains any of the two values in a).

N3 For any set $A = \{x_1, x_2, x_3, x_4, x_5\}$ of five distinct positive integers denote by S_A the sum of its elements, and denote by T_A the number of triples (i, j, k) with $1 \le i < j < k \le 5$ for which $x_i + x_j + x_k$ divides S_A.
Find the largest possible value of T_A.

N4 Dragos, the early ruler of Moldavia, and Maria the Oracle play the following game. Firstly, Maria chooses a set S of prime numbers. Then Dragos gives an infinite sequence $x_1, x_2, \ldots$ of distinct positive integers. Then Maria picks a positive integer M and a prime number p from her set S. Finally, Dragos picks a positive integer N and the game ends. Dragos wins if and only if for all integers $n \ge N$ the number x_n is divisible by p^M; otherwise, Maria wins. Who has a winning strategy if the set S must be: a) finite; b) infinite?

Proposed by *Boris Stanković, Bosnia and Herzegovina*

N5 Find all pairs of integers (x, y) such that $x^2 + 5y^2 = 2021y$.

N6 Given a positive integer $n \geq 2$, we define $f(n)$ to be the sum of all remainders obtained by dividing n by all positive integers less than n. For example dividing 5 with $1, 2, 3$ and 4 we have remainders equal to $0, 1, 2$ and 1 respectively. Therefore $f(5) = 0+1+2+1 = 4$. Find all positive integers $n \geq 3$ such that $f(n) = f(n-1) + (n-2)$.

N7 Alice chooses a prime number $p > 2$ and then Bob chooses a positive integer n_0. Alice, in the first move, chooses an integer $n_1 > n_0$ and calculates the expression $s_1 = n_0^{n_1} + n_1^{n_0}$; then Bob, in the second move, chooses an integer $n_2 > n_1$ and calculates the expression $s_2 = n_1^{n_2} + n_2^{n_1}$; etc. one by one. (Each player knows the numbers chosen by the other in the previous moves.) The winner is the one who first chooses the number n_k such that p divides $s_k(s_1 + 2s_2 + + ks_k)$. Who has a winning strategy?

Proposed by *Borche Joshevski, Macedonia*

C1 In Mathcity, there are infinitely many buses and infinitely many stations. The stations are indexed by the powers of $2 : 1, 2, 4, 8, 16, ...$ Each bus goes by finitely many stations, and the bus number is the sum of all the stations it goes by. For simplifications, the mayor of Mathcity wishes that the bus numbers form an arithmetic progression with common difference r and whose first term is the favourite number of the mayor. For which positive integers r is it always possible that, no matter the favourite number of the mayor, given any m stations, there is a bus going by all of them?

Proposed by *Savinien Kreczman and Martin Rakovsky, France*

C2 Let n be a positive integer. We are given a $3n \times 3n$ board whose unit squares are colored in black and white in such way that starting with the top left square, every third diagonal is colored in black and the rest of the board is in white. In one move, one can take a 2×2 square and change the color of all its squares in such way that white squares become orange, orange ones become black and black ones become white. Find all n for which, using a finite number of moves, we can make all the squares which were initially black white, and all squares which were initially white black.

Proposed by *Boris Stanković and Marko Dimitrić, Bosnia and Herzegovina*

C3 We have a set of 343 closed jars, each containing blue, yellow and red marbles with the number of marbles from each color being at least 1 and at most 7. No two jars have exactly the same contents. Initially all jars are with the caps up. To flip a jar will mean to change its position from cap-up to cap-down or vice versa. It is allowed to choose a
triple of positive integers $(b; y; r) \in \{1; 2; ...; 7\}^3$ and flip all the jars whose number of blue, yellow and red marbles differ by not more than 1 from b, y, r, respectively. After n moves all the jars turned out to be with the caps down. Find the number of all possible values of n, if $n \leq 2021$.

C4 Alice and Bob play a game together as a team on a 100×100 board with all unit squares initially white. Alice sets up the game by coloring exactly k of the unit squares red at the beginning. After that, a legal move for Bob is to choose a row or column with at least 10 red squares and color all of the remaining squares in it red. What is the
smallest k such that Alice can set up a game in such a way that Bob can color the entire board red after finitely many moves?

Proposed by *Nikola Velov, Macedonia*

C5 Let M be a subset of the set of 2021 integers $\{1, 2, 3, ..., 2021\}$ such that for any three elements (not necessarily distinct) a, b, c of M we have $|a + b - c| > 10$.
Determine the largest possible number of elements of M.

C6 Given an $m \times n$ table consisting of mn unit cells. Alice and Bob play the following game: Alice goes first and the one who moves colors one of the empty cells with one of the given three colors. Alice wins if there is a figure, such as the ones below, having three different colors. Otherwise Bob is the winner. Determine the winner for all cases of m
and n where $m, n \geq 3$.

Proposed by *Toghrul Abbasov, Azerbaijan*
