IMO Shortlist 2021

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– Algebra

A1 Let n be a positive integer. Given is a subset A of $\{0, 1, ..., 5^n\}$ with $4n + 2$ elements. Prove that there exist three elements $a < b < c$ from A such that $c + 2a > 3b$.

Proposed by Dominik Burek and Tomasz Ciesla, Poland

A2 Which positive integers n make the equation

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\left\lfloor\frac{ij}{n+1}\right\rfloor = \frac{n^2(n-1)}{4}$$

true?

A3 For each integer $n \geq 1$, compute the smallest possible value of

$$\sum_{k=1}^{n}\left\lfloor\frac{a_k}{k}\right\rfloor$$

over all permutations $(a_1, \ldots, a_n)$ of $\{1, \ldots, n\}$.

Proposed by Shahjalal Shohag, Bangladesh

A4 Show that the inequality

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sqrt{|x_i - x_j|} \leqslant \sum_{i=1}^{n}\sum_{j=1}^{n}\sqrt{|x_i + x_j|}$$

holds for all real numbers $x_1, \ldots x_n$.

A5 Let $n \geq 2$ be an integer and let $a_1, a_2, \ldots, a_n$ be positive real numbers with sum 1. Prove that

$$\sum_{k=1}^{n}\frac{a_k}{1-a_k}(a_1 + a_2 + \cdots + a_{k-1})^2 < \frac{1}{3}.$$

A6 Let $m \geq 2$ be an integer, A a finite set of integers (not necessarily positive) and $B_1, B_2, ..., B_m$ subsets of A. Suppose that, for every $k = 1, 2, ..., m$, the sum of the elements of B_k is m^k. Prove that A contains at least $\frac{m}{2}$ elements.

A7 Let $n \geqslant 1$ be an integer, and let $x_0, x_1, \ldots, x_{n+1}$ be $n+2$ non-negative real numbers that satisfy $x_i x_{i+1} - x_{i-1}^2 \geqslant 1$ for all $i = 1, 2, \ldots, n$. Show that

$$x_0 + x_1 + \cdots + x_n + x_{n+1} > \left(\frac{2n}{3}\right)^{3/2}.$$

Pakawut Jiradilok and Wijit Yangjit, Thailand

A8 Determine all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy

$$(f(a) - f(b))(f(b) - f(c))(f(c) - f(a)) = f(ab^2 + bc^2 + ca^2) - f(a^2b + b^2c + c^2a)$$

for all real numbers a, b, c.

Proposed by Ankan Bhattacharya, USA

– Combinatorics

C1 Let S be an infinite set of positive integers, such that there exist four pairwise distinct $a, b, c, d \in S$ with $\gcd(a, b) \neq \gcd(c, d)$. Prove that there exist three pairwise distinct $x, y, z \in S$ such that $\gcd(x, y) = \gcd(y, z) \neq \gcd(z, x)$.

C2 Let $n \geq 3$ be a fixed integer. There are $m \geq n+1$ beads on a circular necklace. You wish to paint the beads using n colors, such that among any $n+1$ consecutive beads every color appears at least once. Find the largest value of m for which this task is *not* possible.

Carl Schildkraut, USA

C3 Two squirrels, Bushy and Jumpy, have collected 2021 walnuts for the winter. Jumpy numbers the walnuts from 1 through 2021, and digs 2021 little holes in a circular pattern in the ground around their favourite tree. The next morning Jumpy notices that Bushy had placed one walnut into each hole, but had paid no attention to the numbering. Unhappy, Jumpy decides to reorder the walnuts by performing a sequence of 2021 moves. In the k-th move, Jumpy swaps the positions of the two walnuts adjacent to walnut k.

Prove that there exists a value of k such that, on the k-th move, Jumpy swaps some walnuts a and b such that $a < k < b$.

C4 The kingdom of Anisotropy consists of n cities. For every two cities there exists exactly one direct one-way road between them. We say that a [i]path from X to Y[/i] is a sequence of roads such that one can move from X to Y along this sequence without returning to an already visited city. A collection of paths is called *diverse* if no road belongs to two or more paths in the collection.

Let A and B be two distinct cities in Anisotropy. Let N_{AB} denote the maximal number of paths in a diverse collection of paths from A to B. Similarly, let N_{BA} denote the maximal number of paths in a diverse collection of paths from B to A. Prove that the equality $N_{AB} = N_{BA}$ holds if and only if the number of roads going out from A is the same as the number of roads going out from B.

Proposed by Warut Suksompong, Thailand

C5 Let n and k be two integers with $n > k \geqslant 1$. There are $2n+1$ students standing in a circle. Each student S has $2k$ *neighbors* - namely, the k students closest to S on the left, and the k students closest to S on the right.

Suppose that $n+1$ of the students are girls, and the other n are boys. Prove that there is a girl with at least k girls among her neighbors.

Proposed by Gurgen Asatryan, Armenia

C6 A hunter and an invisible rabbit play a game on an infinite square grid. First the hunter fixes a colouring of the cells with finitely many colours. The rabbit then secretly chooses a cell to start in. Every minute, the rabbit reports the colour of its current cell to the hunter, and then secretly moves to an adjacent cell that it has not visited before (two cells are adjacent if they share an edge). The hunter wins if after some finite time either:-the rabbit cannot move; or
-the hunter can determine the cell in which the rabbit started.Decide whether there exists a winning strategy for the hunter.

Proposed by Aron Thomas

C7 Consider a checkered $3m \times 3m$ square, where m is an integer greater than 1. A frog sits on the lower left corner cell S and wants to get to the upper right corner cell F. The frog can hop from any cell to either the next cell to the right or the next cell upwards.

Some cells can be *sticky*, and the frog gets trapped once it hops on such a cell. A set X of cells is called *blocking* if the frog cannot reach F from S when all the cells of X are sticky. A blocking set is *minimal* if it does not contain a smaller blocking set.-Prove that there exists a minimal blocking set containing at least $3m^2 - 3m$ cells.
-Prove that every minimal blocking set containing at most $3m^2$ cells.

C8 Determine the largest integer N for which there exists a table T of integers with N rows and 100 columns that has the following properties: (i) Every row contains the numbers 1, 2, $\ldots$, 100 in some order. (ii) For any two distinct rows r and s, there is a column c such that $|T(r,c) - T(s,c)| \geq 2$. (Here $T(r,c)$ is the entry in row r and column c.)

– Geometry

G1 Let $ABCD$ be a parallelogram with $AC = BC$. A point P is chosen on the extension of ray AB past B. The circumcircle of ACD meets the segment PD again at Q. The circumcircle of triangle APQ meets the segment PC at R. Prove that lines CD, AQ, BR are concurrent.

G2 Let Γ be a circle with centre I, and $ABCD$ a convex quadrilateral such that each of the segments AB, BC, CD and DA is tangent to Γ. Let Ω be the circumcircle of the triangle AIC. The extension of BA beyond A meets Ω at X, and the extension of BC beyond C meets Ω at Z. The extensions of AD and CD beyond D meet Ω at Y and T, respectively. Prove that

$$AD + DT + TX + XA = CD + DY + YZ + ZC.$$

Proposed by Dominik Burek, Poland and Tomasz Ciesla, Poland

G3 Consider a 100×100 square unit lattice **L** (hence **L** has 10000 points). Suppose $\mathcal{F}$ is a set of polygons such that all vertices of polygons in $\mathcal{F}$ lie in **L** and every point in **L** is the vertex of exactly one polygon in $\mathcal{F}$. Find the maximum possible sum of the areas of the polygons in $\mathcal{F}$.

Michael Ren and Ankan Bhattacharya, USA

G4 Let $ABCD$ be a quadrilateral inscribed in a circle Ω. Let the tangent to Ω at D meet rays BA and BC at E and F, respectively. A point T is chosen inside $\triangle ABC$ so that $\overline{TE} \parallel \overline{CD}$ and $\overline{TF} \parallel \overline{AD}$. Let $K \neq D$ be a point on segment DF satisfying $TD = TK$. Prove that lines AC, DT, and BK are concurrent.

G5 Let $ABCD$ be a cyclic quadrilateral whose sides have pairwise different lengths. Let O be the circumcenter of $ABCD$. The internal angle bisectors of $\angle ABC$ and $\angle ADC$ meet AC at B_1 and D_1, respectively. Let O_B be the center of the circle which passes through B and is tangent to $\overline{AC}$ at D_1. Similarly, let O_D be the center of the circle which passes through D and is tangent to $\overline{AC}$ at B_1.

Assume that $\overline{BD_1} \parallel \overline{DB_1}$. Prove that O lies on the line $\overline{O_BO_D}$.

G6 Find all integers $n \geq 3$ for which every convex equilateral n-gon of side length 1 contains an equilateral triangle of side length 1. (Here, polygons contain their boundaries.)

G7 Let D be an interior point of the acute triangle ABC with $AB > AC$ so that $\angle DAB = \angle CAD$. The point E on the segment AC satisfies $\angle ADE = \angle BCD$, the point F on the segment AB satisfies $\angle FDA = \angle DBC$, and the point X on the line AC satisfies $CX = BX$. Let O_1 and O_2 be the circumcenters of the triangles ADC and EXD, respectively. Prove that the lines BC, EF, and O_1O_2 are concurrent.

G8 Let ABC be a triangle with circumcircle ω and let Ω_A be the A-excircle. Let X and Y be the intersection points of ω and Ω_A. Let P and Q be the projections of A onto the tangent lines to Ω_A at X and Y respectively. The tangent line at P to the circumcircle of the triangle APX intersects the tangent line at Q to the circumcircle of the triangle AQY at a point R. Prove that $\overline{AR} \perp \overline{BC}$.

– Number Theory

N1 Find all positive integers $n \geq 1$ such that there exists a pair (a, b) of positive integers, such that $a^2 + b + 3$ is not divisible by the cube of any prime, and

$$n = \frac{ab + 3b + 8}{a^2 + b + 3}.$$

N2 Let $n \geqslant 100$ be an integer. Ivan writes the numbers $n, n+1, \ldots, 2n$ each on different cards. He then shuffles these $n+1$ cards, and divides them into two piles. Prove that at least one of the piles contains two cards such that the sum of their numbers is a perfect square.

N3 Find all positive integers n with the following property: the k positive divisors of n have a permutation $(d_1, d_2, \ldots, d_k)$ such that for $i = 1, 2, \ldots, k$, the number $d_1 + d_2 + \cdots + d_i$ is a perfect square.

N4 Let $r > 1$ be a rational number. Alice plays a solitaire game on a number line. Initially there is a red bead at 0 and a blue bead at 1. In a move, Alice chooses one of the beads and an integer $k \in \mathbb{Z}$. If the chosen bead is at x, and the other bead is at y, then the bead at x is moved to the point x' satisfying $x' - y = r^k(x - y)$.

Find all r for which Alice can move the red bead to 1 in at most 2021 moves.

N5 Show that $n! = a^{n-1} + b^{n-1} + c^{n-1}$ has only finitely many solutions in positive integers.

Proposed by Dorlir Ahmeti, Albania

N6 Determine all integers $n \geqslant 2$ with the following property: every n pairwise distinct integers whose sum is not divisible by n can be arranged in some order $a_1, a_2, \ldots, a_n$ so that n divides $1 \cdot a_1 + 2 \cdot a_2 + \cdots + n \cdot a_n$.

Arsenii Nikolaiev, Anton Trygub, Oleksii Masalitin, and Fedir Yudin

N7 Let $a_1, a_2, a_3, \ldots$ be an infinite sequence of positive integers such that a_{n+2m} divides $a_n + a_{n+m}$ for all positive integers n and m. Prove that this sequence is eventually periodic, i.e. there exist positive integers N and d such that $a_n = a_{n+d}$ for all $n > N$.

N8 Find all positive integers n for which there exists a polynomial $P(x) \in \mathbb{Z}[x]$ such that for every positive integer $m \geq 1$, the numbers $P^m(1), \ldots, P^m(n)$ leave exactly $\lceil n/2^m \rceil$ distinct remainders when divided by n. (Here, P^m means P applied m times.)

Proposed by Carl Schildkraut, USA