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Art of Problem Solving

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– Day 1

1 Let $n \geq 3$ be a fixed integer. There are $m \geq n+1$ beads on a circular necklace. You wish to paint the beads using n colors, such that among any $n+1$ consecutive beads every color appears at least once. Find the largest value of m for which this task is *not* possible.

Carl Schildkraut, USA

2 Let $ABCD$ be a quadrilateral inscribed in a circle Ω. Let the tangent to Ω at D meet rays BA and BC at E and F, respectively. A point T is chosen inside $\triangle ABC$ so that $\overline{TE} \parallel \overline{CD}$ and $\overline{TF} \parallel \overline{AD}$. Let $K \neq D$ be a point on segment DF satisfying $TD = TK$. Prove that lines $AC, DT,$ and BK are concurrent.

3 Determine all integers $n \geqslant 2$ with the following property: every n pairwise distinct integers whose sum is not divisible by n can be arranged in some order $a_1, a_2, \ldots, a_n$ so that n divides $1 \cdot a_1 + 2 \cdot a_2 + \cdots + n \cdot a_n$.

Arsenii Nikolaiev, Anton Trygub, Oleksii Masalitin, and Fedir Yudin

– Day 2

1 Let $r > 1$ be a rational number. Alice plays a solitaire game on a number line. Initially there is a red bead at 0 and a blue bead at 1. In a move, Alice chooses one of the beads and an integer $k \in \mathbb{Z}$. If the chosen bead is at x, and the other bead is at y, then the bead at x is moved to the point x' satisfying $x' - y = r^k(x-y)$.

Find all r for which Alice can move the red bead to 1 in at most 2021 moves.

2 Let $ABCD$ be a cyclic quadrilateral whose sides have pairwise different lengths. Let O be the circumcenter of $ABCD$. The internal angle bisectors of $\angle ABC$ and $\angle ADC$ meet AC at B_1 and D_1, respectively. Let O_B be the center of the circle which passes through B and is tangent to $\overline{AC}$ at D_1. Similarly, let O_D be the center of the circle which passes through D and is tangent to $\overline{AC}$ at B_1.

Assume that $\overline{BD_1} \parallel \overline{DB_1}$. Prove that O lies on the line $\overline{O_BO_D}$.

3 Let $n \geqslant 1$ be an integer, and let $x_0, x_1, \ldots, x_{n+1}$ be $n+2$ non-negative real numbers that satisfy

$x_i x_{i+1} - x_{i-1}^2 \geqslant 1$ for all $i = 1, 2, \ldots, n$. Show that

$$x_0 + x_1 + \cdots + x_n + x_{n+1} > \left(\frac{2n}{3}\right)^{3/2}.$$

Pakawut Jiradilok and Wijit Yangjit, Thailand

– Day 3

1 Which positive integers n make the equation

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \left\lfloor \frac{ij}{n+1} \right\rfloor = \frac{n^2(n-1)}{4}$$

true?

2 The kingdom of Anisotropy consists of n cities. For every two cities there exists exactly one direct one-way road between them. We say that a [i]path from X to Y[/i] is a sequence of roads such that one can move from X to Y along this sequence without returning to an already visited city. A collection of paths is called *diverse* if no road belongs to two or more paths in the collection.

Let A and B be two distinct cities in Anisotropy. Let N_{AB} denote the maximal number of paths in a diverse collection of paths from A to B. Similarly, let N_{BA} denote the maximal number of paths in a diverse collection of paths from B to A. Prove that the equality $N_{AB} = N_{BA}$ holds if and only if the number of roads going out from A is the same as the number of roads going out from B.

Proposed by Warut Suksompong, Thailand

3 Find all integers $n \geq 3$ for which every convex equilateral n-gon of side length 1 contains an equilateral triangle of side length 1. (Here, polygons contain their boundaries.)

– Day 4

1 For each integer $n \geq 1,$ compute the smallest possible value of

$$\sum_{k=1}^{n} \left\lfloor \frac{a_k}{k} \right\rfloor$$

over all permutations $(a_1, \ldots, a_n)$ of $\{1, \ldots, n\}$.

Proposed by Shahjalal Shohag, Bangladesh

2 Show that $n! = a^{n-1} + b^{n-1} + c^{n-1}$ has only finitely many solutions in positive integers.

Proposed by Dorlir Ahmeti, Albania

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3 Consider a checkered $3m \times 3m$ square, where m is an integer greater than 1. A frog sits on the lower left corner cell S and wants to get to the upper right corner cell F. The frog can hop from any cell to either the next cell to the right or the next cell upwards.

Some cells can be *sticky*, and the frog gets trapped once it hops on such a cell. A set X of cells is called *blocking* if the frog cannot reach F from S when all the cells of X are sticky. A blocking set is *minimal* if it does not contain a smaller blocking set.-Prove that there exists a minimal blocking set containing at least $3m^2 - 3m$ cells.
-Prove that every minimal blocking set containing at most $3m^2$ cells.

– Day 5

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1 Find all positive integers n with the following property: the k positive divisors of n have a permutation $(d_1, d_2, \ldots, d_k)$ such that for $i = 1, 2, \ldots, k$, the number $d_1 + d_2 + \cdots + d_i$ is a perfect square.

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2 Let $n \geq 2$ be an integer and let $a_1, a_2, \ldots, a_n$ be positive real numbers with sum 1. Prove that

$$\sum_{k=1}^{n} \frac{a_k}{1-a_k}(a_1 + a_2 + \cdots + a_{k-1})^2 < \frac{1}{3}.$$

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3 A hunter and an invisible rabbit play a game on an infinite square grid. First the hunter fixes a colouring of the cells with finitely many colours. The rabbit then secretly chooses a cell to start in. Every minute, the rabbit reports the colour of its current cell to the hunter, and then secretly moves to an adjacent cell that it has not visited before (two cells are adjacent if they share an edge). The hunter wins if after some finite time either:-the rabbit cannot move; or
-the hunter can determine the cell in which the rabbit started.Decide whether there exists a winning strategy for the hunter.

Proposed by Aron Thomas

– Day 6

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1 Consider a 100×100 square unit lattice **L** (hence **L** has 10000 points). Suppose $\mathcal{F}$ is a set of polygons such that all vertices of polygons in $\mathcal{F}$ lie in **L** and every point in **L** is the vertex of exactly one polygon in $\mathcal{F}$. Find the maximum possible sum of the areas of the polygons in $\mathcal{F}$.

Michael Ren and Ankan Bhattacharya, USA

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2 Let n and k be two integers with $n > k \geqslant 1$. There are $2n+1$ students standing in a circle. Each student S has $2k$ *neighbors* - namely, the k students closest to S on the left, and the k students closest to S on the right.

Suppose that $n+1$ of the students are girls, and the other n are boys. Prove that there is a girl with at least k girls among her neighbors.

Proposed by Gurgen Asatryan, Armenia

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3 Let $a_1, a_2, a_3, \ldots$ be an infinite sequence of positive integers such that a_{n+2m} divides $a_n + a_{n+m}$ for all positive integers n and m. Prove that this sequence is eventually periodic, i.e. there exist positive integers N and d such that $a_n = a_{n+d}$ for all $n > N$.
