Germany Team Selection Test 2022

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–	VAIMO 1
1	Let S be an infinite set of positive integers, such that there exist four pairwise distinct $a, b, c, d \in S$ with $\gcd(a, b) \neq \gcd(c, d)$. Prove that there exist three pairwise distinct $x, y, z \in S$ such that $\gcd(x, y) = \gcd(y, z) \neq \gcd(z, x)$.
2	Let $ABCD$ be a parallelogram with $AC = BC$. A point P is chosen on the extension of ray AB past B. The circumcircle of ACD meets the segment PD again at Q. The circumcircle of triangle APQ meets the segment PC at R. Prove that lines CD, AQ, BR are concurrent.
3	For each integer $n \geq 1$, compute the smallest possible value of $$\sum_{k=1}^{n} \left\lfloor \frac{a_k}{k} \right\rfloor$$ over all permutations $(a_1, \ldots, a_n)$ of $\{1, \ldots, n\}$. *Proposed by Shahjalal Shohag, Bangladesh*
–	VAIMO 2
1	Let n be a positive integer. Given is a subset A of $\{0, 1, ..., 5^n\}$ with $4n + 2$ elements. Prove that there exist three elements $a < b < c$ from A such that $c + 2a > 3b$. *Proposed by Dominik Burek and Tomasz Ciesla, Poland*
2	Find all positive integers $n \geq 1$ such that there exists a pair (a, b) of positive integers, such that $a^2 + b + 3$ is not divisible by the cube of any prime, and $$n = \frac{ab + 3b + 8}{a^2 + b + 3}.$$
3	Consider a 100×100 square unit lattice **L** (hence **L** has 10000 points). Suppose $\mathcal{F}$ is a set of polygons such that all vertices of polygons in $\mathcal{F}$ lie in **L** and every point in **L** is the vertex of exactly one polygon in $\mathcal{F}$. Find the maximum possible sum of the areas of the polygons in $\mathcal{F}$. *Michael Ren and Ankan Bhattacharya, USA*

–	AIMO 1

1 Let $n \geq 2$ be an integer and let $a_1, a_2, \ldots, a_n$ be positive real numbers with sum 1. Prove that

$$\sum_{k=1}^{n} \frac{a_k}{1-a_k}(a_1+a_2+\cdots+a_{k-1})^2 < \frac{1}{3}.$$

2 Find all integers $n \geq 3$ for which every convex equilateral n-gon of side length 1 contains an equilateral triangle of side length 1. (Here, polygons contain their boundaries.)

3 Find all positive integers n with the following property: the k positive divisors of n have a permutation $(d_1, d_2, \ldots, d_k)$ such that for $i = 1, 2, \ldots, k$, the number $d_1 + d_2 + \cdots + d_i$ is a perfect square.

–	AIMO 2

1 Let $a_1, a_2, \ldots, a_n$ be n positive integers, and let $b_1, b_2, \ldots, b_m$ be m positive integers such that $a_1 a_2 \cdots a_n = b_1 b_2 \cdots b_m$. Prove that a rectangular table with n rows and m columns can be filled with positive integer entries in such a way that

* the product of the entries in the i-th row is a_i (for each $i \in \{1, 2, \ldots, n\}$);

* the product of the entries in the j-th row is b_j (for each $i \in \{1, 2, \ldots, m\}$).

2 Given two positive integers n and m and a function $f : \mathbb{Z} \times \mathbb{Z} \to \{0, 1\}$ with the property that

$$f(i,j) = f(i+n,j) = f(i,j+m) \qquad \text{for all } (i,j) \in \mathbb{Z} \times \mathbb{Z}.$$

Let $[k] = \{1, 2, \ldots, k\}$ for each positive integer k.
Let a be the number of all $(i,j) \in [n] \times [m]$ satisfying

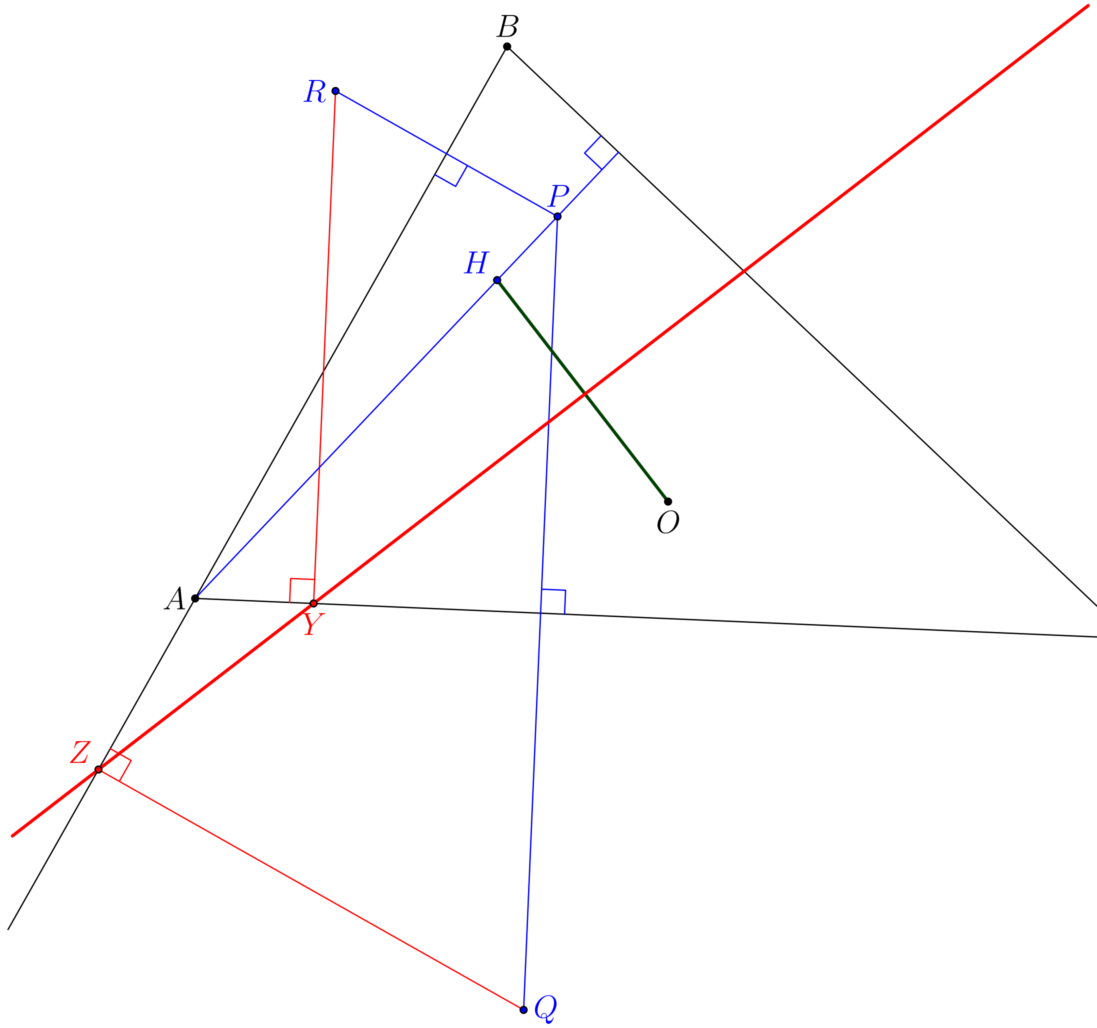
$$f(i,j) = f(i+1,j) = f(i,j+1).$$

Let b be the number of all $(i,j) \in [n] \times [m]$ satisfying

$$f(i,j) = f(i-1,j) = f(i,j-1).$$

Prove that $a = b$.

3 Let ABC be a triangle with orthocenter H and circumcenter O. Let P be a point in the plane such that $AP \perp BC$. Let Q and R be the reflections of P in the lines CA and AB, respectively. Let Y be the orthogonal projection of R onto CA. Let Z be the orthogonal projection of Q onto AB. Assume that $H \neq O$ and $Y \neq Z$. Prove that $YZ \perp HO$.



– AIMO 3

2 Let $ABCD$ be a cyclic quadrilateral whose sides have pairwise different lengths. Let O be the circumcenter of $ABCD$. The internal angle bisectors of $\angle ABC$ and $\angle ADC$ meet AC at B_1 and D_1, respectively. Let O_B be the center of the circle which passes through B and is tangent to $\overline{AC}$ at D_1. Similarly, let O_D be the center of the circle which passes through D and is tangent to $\overline{AC}$ at B_1.

Assume that $\overline{BD_1} \parallel \overline{DB_1}$. Prove that O lies on the line $\overline{O_BO_D}$.

3 Consider a checkered $3m \times 3m$ square, where m is an integer greater than 1. A frog sits on the lower left corner cell S and wants to get to the upper right corner cell F. The frog can hop from any cell to either the next cell to the right or the next cell upwards.

Some cells can be *sticky*, and the frog gets trapped once it hops on such a cell. A set X of cells is called *blocking* if the frog cannot reach F from S when all the cells of X are sticky. A blocking set is *minimal* if it does not contain a smaller blocking set.-Prove that there exists a minimal blocking set containing at least $3m^2 - 3m$ cells.
-Prove that every minimal blocking set containing at most $3m^2$ cells.

– AIMO 4

2 Let $r > 1$ be a rational number. Alice plays a solitaire game on a number line. Initially there is a red bead at 0 and a blue bead at 1. In a move, Alice chooses one of the beads and an integer $k \in \mathbb{Z}$. If the chosen bead is at x, and the other bead is at y, then the bead at x is moved to the point x' satisfying $x' - y = r^k(x - y)$.

Find all r for which Alice can move the red bead to 1 in at most 2021 moves.

3 A hunter and an invisible rabbit play a game on an infinite square grid. First the hunter fixes a colouring of the cells with finitely many colours. The rabbit then secretly chooses a cell to start in. Every minute, the rabbit reports the colour of its current cell to the hunter, and then secretly moves to an adjacent cell that it has not visited before (two cells are adjacent if they share an edge). The hunter wins if after some finite time either:-the rabbit cannot move; or
-the hunter can determine the cell in which the rabbit started.Decide whether there exists a winning strategy for the hunter.

Proposed by Aron Thomas

– AIMO 5

2 Let $ABCD$ be a quadrilateral inscribed in a circle Ω. Let the tangent to Ω at D meet rays BA and BC at E and F, respectively. A point T is chosen inside $\triangle ABC$ so that $\overline{TE} \parallel \overline{CD}$ and $\overline{TF} \parallel \overline{AD}$. Let $K \neq D$ be a point on segment DF satisfying $TD = TK$. Prove that lines $AC, DT,$ and BK are concurrent.

– AIMO 6

1 Given a triangle ABC and three circles x, y and z such that $A \in y \cap z$, $B \in z \cap x$ and $C \in x \cap y$.

The circle x intersects the line AC at the points X_b and C, and intersects the line AB at the points X_c and B.
The circle y intersects the line BA at the points Y_c and A, and intersects the line BC at the points Y_a and C.
The circle z intersects the line CB at the points Z_a and B, and intersects the line CA at the points Z_b and A.
(Yes, these definitions have the symmetries you would expect.)

Prove that the perpendicular bisectors of the segments Y_aZ_a, Z_bX_b and X_cY_c concur.

2 The kingdom of Anisotropy consists of n cities. For every two cities there exists exactly one direct one-way road between them. We say that a [i]path from X to Y[/i] is a sequence of roads such that one can move from X to Y along this sequence without returning to an already visited city. A collection of paths is called *diverse* if no road belongs to two or more paths in the collection.

Let A and B be two distinct cities in Anisotropy. Let N_{AB} denote the maximal number of paths in a diverse collection of paths from A to B. Similarly, let N_{BA} denote the maximal number of paths in a diverse collection of paths from B to A. Prove that the equality $N_{AB} = N_{BA}$ holds if and only if the number of roads going out from A is the same as the number of roads going out from B.

Proposed by Warut Suksompong, Thailand

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3 Determine all integers $n \geqslant 2$ with the following property: every n pairwise distinct integers whose sum is not divisible by n can be arranged in some order $a_1, a_2, \dots, a_n$ so that n divides $1 \cdot a_1 + 2 \cdot a_2 + \cdots + n \cdot a_n$.

Arsenii Nikolaiev, Anton Trygub, Oleksii Masalitin, and Fedir Yudin

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– AIMO 7

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1 Which positive integers n make the equation

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \left\lfloor \frac{ij}{n+1} \right\rfloor = \frac{n^2(n-1)}{4}$$

true?

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2 Let n and k be two integers with $n > k \geqslant 1$. There are $2n+1$ students standing in a circle. Each student S has $2k$ *neighbors* - namely, the k students closest to S on the left, and the k students closest to S on the right.

Suppose that $n+1$ of the students are girls, and the other n are boys. Prove that there is a girl with at least k girls among her neighbors.

Proposed by Gurgen Asatryan, Armenia

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3 Show that $n! = a^{n-1} + b^{n-1} + c^{n-1}$ has only finitely many solutions in positive integers.

Proposed by Dorlir Ahmeti, Albania
